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GROUPS OF THE FUNDAMENTAL OPERATIONS OF ARITHMETIC

By G. A. Miller

By adding n to x_1, then this sum to x_1, etc., there results an infinite series of numbers*, n, $n + x_1$, $n + 2x_1$, $\cdots$. If these numbers are taken with respect to a given modulus (m) only m/d of them are distinct, d being the highest common factor of m and x_1. These distinct numbers are of the form $n + kd$, $k = 1, 2, \cdots, m/d$. If n_0 represents the smallest one with respect to modulus m, their least positive residues are $n_0 + kd$, $k = 1, 2, \cdots, m/d$. These elementary principles of number theory will be frequently employed in the following developments.

Instead of adding n to one number we may add it to a series of numbers $x_1, x_2, \cdots, x_l$. If the sums thus obtained are again added to this series in any order, and these operations are repeated, the resulting numbers again belong to m/d distinct classes with respect to modulus m, d being the highest common factor of $x_1, x_2, \cdots, x_l, m$. Their least positive residues are again given by $n_0 + kd$, $k = 1, 2, \cdots, m/d$, where n_0 is the smallest number with respect to modulus m that is obtained by the given operations.

As the definitions of a group clearly apply to these operations and as all the numbers are obtained by a repetition of the same operation, it is convenient to say that *adding to one or more given numbers and taking the results with respect to a fixed modulus gives rise to the cyclic group whose order is equal to the number of least positive residues of all the resulting numbers.* For instance, when $n = 8$, $x_1 = 15$, $m = 20$, the least positive residues of the resulting numbers are 3, 8, 13, 18. These become the cyclic group of order 4 when they are increased successively by some multiple of $d = 5$. These results

* All the numbers employed will be regarded as positive integers unless the contrary is stated.

remain unchanged by adding $x_2 = 35$ to the given conditions, but if $x_2 = 12$ the cyclic group becomes of order 20 since $d = 1$ in this case.

These considerations are so elementary that they offer little interest in themselves. They are explicitly stated, partly for the sake of completeness and partly on account of their fundamental importance in the following developments. It should be carefully observed that we add only to $x_1, x_2, \cdots, x_l$. If the resulting numbers were added together, the order of the cyclic group would be m/d', d' being the highest common factor of d and n. The fundamental thought in what precedes is that *the groups of addition are cyclic.* We proceed to consider subtraction. It will appear that its groups are of the dihedral rotation type.

1. Groups of subtraction. If we subtract n from x_1 the resulting number $(x_1 - n)$ may be supposed to take the place of either x_1 or n. In the former case subtraction is an operation of infinite period unless we consider a modulus; in the latter case it is of period two, since $x_1 - (x_1 - n) = n$. We shall here adopt the latter point of view and hence regard subtraction from x_1 as a reflection on the point $x_1/2$, the numbers being represented in the usual manner by points on a line.

Subtracting n from x_1 and then subtracting the resulting number from x_2 is equivalent to reflecting n on the point $x_1/2$ and then reflecting $x_1 - n$ on the point $x_2/2$. The result is equal to a translation through $x_2 - x_1$; that is, n becomes $n + (x_2 - x_1)$. It was seen above that these translations with respect to modulus m generate the cyclic group of order m/d, d being the highest common factor of $x_2 - x_1$ and m. Hence the operations of subtracting from x_1 and x_2 generate the dihedral rotation group of order $2m/d$.* One of the most interesting examples along this line is furnished by taking the complement and the supplement of an angle. As these two operations are equivalent to a rotation through $\pi/2$ they generate the group of order 8 which is identical with the group of movements of the square.† It contains two operators of period 4, five of period 2, and the identity.

While the general theory of the group obtained by subtracting from two numbers follows directly from the theory of the groups which are generated

* *Bulletin of the American Mathematical Society*, vol. 7 (1901), p. 424.

† Davis, *Nebraska University Studies*, vol. 4 (1904), p. 231.

by two operators of order two, yet it seems desirable to add a few details with respect to the special problem on hand. A reflection followed by a translation is equivalent to a reflection of the original point on the point obtained by moving the original point of reflection through half the translation. Hence a reflection followed by a translation is always of period two, and a reflection followed by a translation followed by the same reflection is the inverse of the translation. In other words a translation is transformed into its inverse by any reflection. Hence the order of the group generated by *any* translation and a reflection is twice the order of the group generated by the translation alone. When the translation is equal to two given reflections the group generated by the translation and one of these reflections includes the other reflection.

When n is subtracted from a series of numbers $x_1, x_2, \cdots, x^l$ and the remainders are again subtracted from these numbers, in any order, these operations of order 2 generate the dihedral rotation group of order $2m/d$, where d is the highest common factor of m, and of all the differences of the form $x_\alpha - x_\beta$. This follows immediately from the fact that the translations which are equivalent to the pairs of these reflections generate the cyclic group of order m/d. As there are just two positive points less than m with respect to which all reflections are equivalent modulo m, there are just $2m/d$ different values of n modulo m which go into less than $2m/d$ distinct numbers by the operations of the group of subtraction. Each of these special values goes into m/d distinct values, differing by multiples of d, under this group; *i. e.* these $2m/d$ special values constitute two sets such that all of those of one set go into each other under the group.* Congruent numbers modulo m go into congruent numbers under all of these operations.

2. Groups of multiplication and groups of division. Since the groups of multiplication have received considerable attention † it seems necessary to do little more here than state those results which are fundamental in the study of the groups of division. There is, however, one impor-

* For example, by taking the complement and the supplement we obtain 8 distinct angles from a given angle provided it is not a multiple of $\pi/4$. In the latter case only 4 distinct angles, differing by $\pi/2$, are obtained by these operations.

† *Cf.* ANNALS OF MATHEMATICS, ser. 2, vol. 2 (1901), p. 77.

tant principle connected with these groups which has not received sufficient attention; viz. the *necessary* condition that a set of numbers becomes a group when they are combined according to multiplication. It has been observed that a sufficient condition is the fact that the set is composed of all the numbers which are less than a given modulus (m) and prime to m. That this condition is not necessary follows directly from the fact that the four numbers 2, 4, 6, 8 become a group when their products are taken with respect to modulus 10.

The necessary and sufficient condition that a given set of numbers, which are distinct with respect to modulus m, generates a group with respect to multiplication is that the highest common factor of the modulus and a number of the set is independent of the choice of this number, and that m divided by this highest common factor is prime to this factor. That this condition is sufficient follows directly from the fact that by dividing by the highest common factor we obtain a set of numbers each of which is prime to the new modulus. The condition is also necessary to insure that every number of the set can be obtained by combining any other number of the set with numbers of the set. For example, 5, 35 generate the four-group 5, 25, 35, 55 mod 60; while 5, 25 generate the group of order 2 with respect to the same modulus. In both of these groups 25 corresponds to the identity. In general, the number which is congruent to unity, modulo m divided by the common factor, corresponds to the identity.

When the elements of a group are not prime to the modulus the group is identical with the one obtained when these elements and the modulus are divided by their highest common factor. In other words, all multiplication groups can be represented by elements which are prime to the modulus. Hence this special class of abelian groups is identical with the groups and the sub-groups of the groups of isomorphisms of cyclic groups, whose properties may be assumed to be known.* We proceed to the groups of division and shall find that these are obtained by simply adding operators of period two to the groups of multiplication.

The period of the operation of division as that of subtraction depends upon the point of view. In each case the group may be regarded as the same as

*ANNALS OF MATHEMATICS, *loc. cit.*; LUCAS, *Théorie des nombres*, 1891, p. 395.

that of the corresponding direct operation. That is, we may regard subtraction as an operation on the minuend so that the remainders become new minuends. From this standpoint there is no difference between the group of subtraction and that of addition. Similarly, if we regard the quotients as new dividends there is no essential difference between the group of division and that of multiplication. If, however, the quotient is regarded as a new divisor, the operation of division is always of period two, instead of being of a finite period only when a modulus is employed. In what follows division will be regarded as of period two.

Geometrically the operation of dividing x_1 by n may be regarded as an inversion of n followed by a stretching in the ratio x_1. That is, any point n goes into the point x_1/n by this operation. * It is geometrically evident that this operation is of period two. By dividing two distinct numbers x_1, x_2 there result two operations of period two whose group will now be considered. The product of these two operations, in order, brings n into $(x_2/x_1)n$. If each of the numbers x_1, x_2 is prime to m there will be just one number (x_3) less than m such that $x_2 \equiv x_1 x_3 \bmod m$.† We shall regard x_2/x_1 and x_3 as equal with respect to modulus m. Hence the powers of x_2/x_1 constitute a cyclic group modulo m and the two operations x_1/n, x_2/n generate a dihedral rotation group whose order is twice the order of this cyclic group.

The general case of dividing the l distinct numbers $x_1, x_2, \cdots, x_l$ by n presents scarcely any new considerations. If m is prime to all the numbers $x_1, x_2, \cdots, x_l$ the group (G) generated by these l operations of period two is obtained by adding operations of order two to the group (H) of multiplication generated by the quotients x_1/x_2, $x_1/x_3, \cdots, x_2/x_3, \cdots, x_{l-1}/x_l$. The order of G is twice the order of H, and the additional operations of order two transform each operator of H into its inverse. That is, this group consists of an abelian group, which is found in the group of order $\phi(m)$ generated by all the numbers less than m and prime to m, and as many more operators of period two as there are in this abelian group.

If one of the numbers $x_1, x_2, \cdots, x_l$ occurs among the quotients x_1/x_2,

* This clearly remains true even when the numbers are complex.

† This is also true when m, x_1 and m, x_2 have the same highest common factor (d) and m/d is prime to d, as is evident from the preceding developments. For brevity we shall not mention this evident extension in what follows.

$x_1/x_3, \ldots, x_2/x_3, \ldots, x_{l-1}/x_l$ then all of them are found among these quotients. Hence the coefficients of n and $1/n$ involve either g or $g/2$ distinct numbers modulo m, g being the order of G, and n goes into g or $g/2$ distinct values under G as x_1 does not or does belong to H. For instance, by dividing 2 and 6 by n modulo 11 the following ten distinct coefficients result:* n, $4n$, $5n$, $9n$, $3n$, $\frac{2}{n}$, $\frac{2}{4n}$, $\frac{2}{5n}$, $\frac{2}{9n}$, $\frac{2}{3n}$. No new numbers are obtained by dividing 6 by these numbers since $\frac{6}{n} = \frac{2}{4n}$, $\frac{6}{4n} = \frac{7}{n} = \frac{2}{5n}$, etc. For special values of n these numbers become a group modulo m only when n is equal to one of the given coefficients. The order of this group is equal to the number of distinct coefficients. The prominent thought of this section is that the group of division is obtained from the corresponding group of multiplication in the same way as the group of subtraction is derived from the corresponding group of addition.

3. Groups of subtraction and division. Since the operations of dividing x_1 by n and of subtracting n from x_1 (x_1/n, $x_1 - n$) are of period two, it is of interest to inquire into the group generated by them. As is well known the six values of the anharmonic ratio are obtained from any one of them by a repetition of these two operations when $x_1 = 1$.† Similarly, each of the six trigonometric functions ($\sin^2 x$, $\cos^2 x$, $-\tan^2 x$, $-\cot^2 x$, $\sec^2 x$, $\csc^2 x$) is obtained from any one of them by the same operations. That is, the operations of inversion and subtraction from unity generate the symmetric group of order six. Since these two operations performed in succession are equivalent to an operation of period three, this result follows directly from the theory of the dihedral rotation groups.

When $x_1 = 2$, the given operations lead to another well known group; viz., the group of movements of the square. This follows directly from the fact that the two successive operations $2/n$ and $2 - n$ are equivalent to the operation $\frac{2}{2-n}$ which is of period 4. In fact, $\left(\frac{2}{2-n}\right)^2 = 2 \Big/ \left(2 - \frac{2}{2-n}\right) = \frac{2-n}{1-n}$, and $\left(\frac{2-n}{1-n}\right)^2 = n$. When $x_1 = 3$, there results the dihedral group

* For convenience the unknown (n) is also given.

†*Cf.* Burnside, *Theory of groups of finite order*, 1897, p. 18.

of order 12 since $\frac{3}{3-n}$ is of period six.* It is easy to prove that this group is of infinite order for every larger value of x_1.

The last statement may be proved as follows: Let $s_1 = x_1 - n$, $s_2 = \frac{x_1}{n}$. Then will $s_1 s_2 = x_1 - \frac{x_1}{n}$, $(s_1 s_2)^2 = \frac{x_1 - (x_1 - 1)n}{1 - n}$. In general let $(s_1 s_2)^k = \frac{\alpha + \beta n}{\gamma + \delta n}$. Then will $(s_1 s_2)^{k+1} = \frac{\beta x_1 - (\alpha + x_1\beta)n}{\delta x_1 - (\gamma + \delta x_1)n} = \frac{\alpha' + \beta' n}{\gamma' + \delta' n}$. If the value of each of the fractions β/α, δ/γ lies between $-\frac{1}{2}$ and -2 and if $x_1 \geqq 4$ then will the fractions $\frac{\beta'}{\alpha'}$, $\frac{\delta'}{\gamma'}$, satisfy the same conditions. As these conditions are satisfied in $(s_1 s_2)^2$, they must be satisfied for the higher powers. That is, the two operations in question generate a group of infinite order whenever $x_1 \geqq 4$. For negative values of x_1 the ratios β'/δ', δ'/γ' are negative and not zero whenever β/δ, δ/γ satisfy these conditions. Hence $s_1 s_2$ is also of an infinite period for all negative values of x_1.

From the same substitutions it follows also directly that $s_1 s_2$ is of an infinite period whenever x_1 is a proper fraction; for, if α,β involve the same highest power of x_1 and if the coefficient of this power is unity in each of these terms then will the same be true of α', β'. It is evident that these remarks apply equally to γ, δ and γ', δ'. It should be observed that the groups of this section are independent of any modulus while all of those which precede, with exception of the trivial group of order two, are dependent upon a modulus.

* These examples are special cases of the following general theorems: whenever $x_1^2 = x_2$ the operations of substracting n from x_1 and dividing x_2 by n give rise to the symmetric group of order 6; when $x_1^2 = 2x_2$ these operations lead to the octic group, and when $x_1^2 = 3x_2$ they generate the dihedral rotation group of order 12. When $x_1 = 0$ while x_2 has any other given value, there results the group of division and change of sign. This is evidently the well known four-group. When x_1, x_2 do not satisfy any of these conditions, the two given operations cannot generate a finite group which transforms every rational point of the plane into a rational point. Hence these four-groups are of fundamental importance in the theory of point transformations of the plane. The examples given above illustrate these groups when $x_1 = x_2$.

While it is well known that the same group frequently presents itself in many different investigations, yet it may be in place to emphasize the fact that the octic group is the group of taking the supplement and the complement of an angle, of dividing 2 by n and subtracting n from 2, of the movements of solid space which transform a square into itself, of the three valued function in four letters, of the substitutions in four letters which transform the cyclic group in these letters into itself, of isomorphisms of itself, of subtracting n from 3 and 6 modulo 12, of dividing 1 and 2 by n modulo 5, etc. The number of such illustrations is always infinite but in this case some seem to be of especial interest.

LINEAR DIFFERENTIAL EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

By Maxime Bôcher

If we consider the problem of determining the simple harmonic vibrations of a stretched wire whose cross section changes abruptly at some points, or the analogous problem of the flow of heat in a rod of similar character, we are led to a linear differential equation of the second order whose coefficients are discontinuous functions of the real variable x. Such "finite discontinuities," or even much more complicated ones,* in the coefficients of the differential equation should not be regarded as being in any true sense singular points of the equation, the only peculiarity which they produce in the solutions being a discontinuity in the second derivative. The same is true when the coefficients of the differential equation have discontinuities where they fail to remain finite, provided merely that the integrals of the absolute values of the coefficients converge when extended up to such points of discontinuity. The proofs of the fundamental theorems concerning linear differential equations can be given in such a form as to cover all these cases, and, a fact to which I wish especially to call attention, this greater generality does not involve any material increase in complexity or difficulty of the proof. In the following pages I have given such a treatment, confining myself for the sake of simplicity to equations of the second order though the method used admits of immediate and obvious extension to linear differential equations of any order.

1. General preliminaries. We shall be concerned exclusively with real values of x which lie in the interval

$$(I) \qquad a \leqq x \leqq b.$$

While some of the functions with which we shall deal are continuous throughout (I), others are of a more general character. These we designate as *integrable* according to the

* As for instance the discontinuity which $\sin 1/x$ has at the point $x = 0$.

DEFINITION. *A function $f(x)$ is said to be integrable throughout (I) if it is continuous except at a finite number of points, and if $\int f(x)dx$ converges when extended over any portion of (I).**

The following facts concerning integrable functions are well known, or can be easily proved:

1) If $\phi(x)$ has only a finite number of discontinuities and $|\phi(x)|$ is integrable, then $\phi(x)$ is integrable. If this is the case we shall speak of ϕ as *absolutely integrable*.

2) If $\phi_1, \phi_2, \cdots \phi_k$ are integrable (or absolutely integrable), the function $\phi_1 + \phi_2 + \cdots + \phi_k$ is integrable (or absolutely integrable).

3) If $\phi(x)$ is integrable, $\int_c^x \phi(x)dx$ is continuous.

4) If $\phi(x)$ is absolutely integrable and $f(x)$ is continuous, $f\cdot\phi$ is absolutely integrable.

5) If $\phi(x)$ is integrable and $f(x)$ is continuous and has a continuous (or absolutely integrable) derivative, $f\cdot\phi$ is integrable. (Proof by integration by parts.)

It is in no way necessary, except in §5, to require that the functions with which we deal be real, although, as has been said, the independent variable x is supposed real. By slight and obvious changes the whole treatment can be carried over to the case of analytic functions of complex variables. This has been indicated briefly in a simple case in §4, where *one* of the independent variables (namely λ) is allowed to become complex. Only slight changes would be necessary in order to allow x also to become complex.

2. The Existence Theorem.† We will consider the differential equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x), \tag{1}$$

*It should be carefully noticed that this definition is more general than Riemann's in that it does not require the function to remain finite in (I), but less general in that it admits only a finite number of points of discontinuity.

†*Cf.* the earlier papers of the writer: *Bulletin of the American Mathematical Society*, March, 1899; *American Journal of Mathematics*, vol. 24, 1902, where this same question is treated by slightly different methods.

and assume that r is an integrable function throughout (I) and p and q are absolutely integrable there.*

The question now arises what we should understand by a solution of (1), the difficulty being that we can clearly not demand that a function which satisfies (1) at other points should also satisfy it when one or more of the functions p, q, r become discontinuous. If we make no demand whatever on the solution at such a point of discontinuity of the coefficients of (1), there will clearly be no connection between the values of the solution on the two sides of this point. It is desirable for most purposes to demand that the solution itself and its first derivative be continuous at such points. Thus we lay down the following

DEFINITION. *By a solution of* (1) *is understood a function of x which at every point of* (I) *is continuous and has a continuous first derivative, and at every point of* (I) *where p, q, r are continuous has a second derivative and satisfies* (1).

We now establish the existence theorem:

If c is any point of (I) *and γ, γ_1 any constants, there exists a solution of* (1) *which satisfies the auxiliary conditions*

$$y(c) = \gamma, \quad y'(c) = \gamma_1. \tag{2}$$

Let us establish this theorem first in the special case in which p and q vanish at every point of (I). The general solution of (1) is then

$$y = \int_c^x \left(\int_c^x r(x)dx \right) dx + A(x - c) + B,$$

a function which is obviously continuous and has a continuous first derivative throughout (I). If we let $A = \gamma_1$, $B = \gamma$, conditions (2) are satisfied. Thus the theorem is proved in this special case. The expression for y which we have obtained can be thrown into a more convenient form by an integration by parts, namely:

$$y = x \int_c^x r(x)dx - \int_c^x xr(x)dx + \gamma_1(x - c) + \gamma.$$

* It is not unlikely that it would be sufficient to require that p and q be merely integrable. The writer would be glad to learn of a method by which this more general case could be treated.

By a slight change of notation we get the following expression alongside of which we write for reference the value of y':

$$(3)\qquad \begin{cases} y = \int_c^x (x-\xi) r(\xi)\, d\xi + \gamma_1 (x-c) + \gamma, \\ y' = \int_c^x r(\xi)\, d\xi + \gamma_1. \end{cases}$$

We now treat the general case by the method of successive approximations. Starting from the value $y_0 = 0$, we compute the successive approximations $y_1, y_2, \cdots$ by means of the recurrent relation:

$$y_n'' = -p y_{n-1}' - q y_{n-1} + r \qquad (n = 1, 2, \cdots),$$

and the auxiliary conditions

$$y_n(c) = \gamma \quad y_n'(c) = \gamma_1.$$

We shall prove that y_n approaches a limit as n becomes infinite, and that this limit is the solution of (1) whose existence we wish to establish.

If we write

$$u_n(x) = y_{n+1}(x) - y_n(x),$$

y_n is obviously the sum of the first n terms of the series

$$(4)\qquad u_0(x) + u_1(x) + u_2(x) + \cdots.$$

We must then prove that this series converges and represents a solution of (1) which satisfies conditions (2). It is convenient to consider by the side of (4) the series

$$(5)\qquad u_0'(x) + u_1'(x) + u_2'(x) + \cdots.$$

Since u_0 and y_1 are identical, we see that u_0 and u_0' are given directly by formulæ (3). The subsequent u's may be determined by the recurrent relation

$$u_n'' = -p(x)\, u_{n-1}' - q(x) u_{n-1} \qquad (n = 1, 2, \cdots)$$

with the auxiliary conditions

$$u_n(c) = 0, \quad u_n'(c) = 0,$$

as we see by subtracting from one another the recurrent relations and also

the auxiliary conditions for determining y_{n+1} and y_n. Applying (3) we now find

$$(6)\qquad \begin{cases} u_n(x) = -\int_c^x (x-\xi)\Big(p(\xi)u'_{n-1}(\xi) + q(\xi)u_{n-1}(\xi)\Big)d\xi, \\ u'_n(x) = -\int_c^x \Big(p(\xi)u'_{n-1}(\xi) + q(\xi)u_{n-1}(\xi)\Big)d\xi. \end{cases}$$

Thus we see that all the terms of (4) and (5) are continuous functions of x throughout (I).

Let us now introduce a positive constant l subject to the conditions

$$(7)\qquad l \geqq 1, \qquad l \geqq b - a.$$

We then infer from (6) the inequality

$$(8)\qquad \left.\begin{matrix} |u_n(x)| \\ |u'_n(x)| \end{matrix}\right\} \leqq l\int_c^x \Big(|p(\xi)|+|q(\xi)|\Big)\Big(|u'_{n-1}(\xi)|+|u_{n-1}(\xi)|\Big)|d\xi| \qquad (n = 1, 2, \cdots).$$

Since $u_0(x)$ and $u'_0(x)$ are continuous throughout (I), there exists a positive constant C such that

$$(9)\qquad |u_0(x)| \leqq C, \qquad |u'_0(x)| \leqq C.$$

We are now in a position to prove the fundamental inequalities

$$(10)\qquad \left.\begin{matrix} |u_n(x)| \\ |u'_n(x)| \end{matrix}\right\} \leqq C\,\frac{\Big[2l\int_c^x \Big(|p(\xi)|+|q(\xi)|\Big)|d\xi|\Big]^n}{n!}.$$

Since (10) reduces to (9) when $n = 0$, it is sufficient to prove that if it is true when $n = k$, it is true when $n = k + 1$.

Letting $n = k + 1$ in (8), and replacing $|u_k(\xi)|$ and $|u_k'(\xi)|$ in the second member of (8) by values obtained by letting $n = k$ in (10), we get

$$\left.\begin{matrix} |u_{k+1}(x)| \\ |u'_{k+1}(x)| \end{matrix}\right\} \leqq C\,\frac{(2l)^{k+1}}{k!}\int_c^x \Big(|p(\xi)|+|q(\xi)|\Big)\Big[\int_c^\xi \Big(|p(\xi)|+|q(\xi)|\Big)|d\xi|\Big]^k |d\xi|.$$

From this we deduce the special case of (10) which we wish to establish by making use of the formula

$$\int_c^x \phi(\xi)\Big(\int_c^\xi \phi(\xi)d\xi\Big)^k d\xi = \frac{1}{k+1}\Big(\int_c^x \phi(\xi)d\xi\Big)^{k+1},$$

whose correctness follows by an integration by parts.

If now we let

$$M = \int_a^b \Big(|p(\xi)| + |q(\xi)| \Big) d\xi,$$

we obtain from (10) the simpler inequalities

$$\left.\begin{matrix} |u_n(x)| \\ |u_n'(x)| \end{matrix}\right\} \leqq C \frac{(2lM)^n}{n!}. \tag{11}$$

These inequalities show that at every point of (I) the terms of the serie (4) and (5) do not exceed in absolute value the terms of the series

$$C + C\frac{2lM}{1!} + C\frac{(2lM)^2}{2!} + \cdots,$$

and, since this is a convergent series of positive constant terms, it follows by Weierstrass's fundamental test that the series (4) and (5) are uniformly convergent throughout (I), and therefore, since their terms are continuous, represent continuous functions throughout this interval. Moreover, by a well known theorem concerning uniform convergence, we see that (5) represents the derivative of (4).

Denoting by $y(x)$ the function represented by (4), it is clear that y satisfies conditions (2). In order to prove that it is the solution of (1) whose existence we wish to establish it is then merely necessary to prove that y satisfies (1) at every point of (I) where p, q, r are continuous.

Let x_0 be any such point, and surround it by an interval (J) lying within (I) throughout which p, q, r are continuous. If we multiply (5) by $-p(x)$, (4) by $-q(x)$ and add the two resulting series together, we get, as we see by referring to the differential equation for u_n,

$$-p(x)\,y'(x) - q(x)\,y(x) = u_1''(x) + u_2''(x) + \cdots. \tag{12}$$

From the way in which this series was obtained it is clear that it is uniformly convergent throughout (J) and that its terms are continuous there. Accordingly, since it may also be obtained by differentiating (5) term by term (the first term being omitted) its value is

$$y''(x) - u_0''(x) = y''(x) - r(x).$$

Substituting this value in (12), we see that y satisfies (1) throughout (J).

That the solution whose existence we have established is unique is stated by the following theorem which we now proceed to prove:

If two solutions of (1) *satisfy conditions* (2), *they are identically equal throughout* (*I*).

The difference of two such solutions is a solution of the homogeneous equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0, \tag{13}$$

satisfying the auxiliary conditions

$$y(c) = 0, \qquad y'(c) = 0. \tag{14}$$

Hence our theorem will be established if we can prove the following more special result:

A solution y_1 *of* (13), *which satisfies the auxiliary conditions* (14) *vanishes at every point of* (*I*).

We prove this by a method due to Sturm.* Let x_0 be any point of (*I*), and consider a solution y_2 of (13) satisfying the auxiliary conditions

$$y_2(x_0) = 0, \quad y'_2(x_0) = 1.$$

By Abel's formula

$$W(x) = y_1(x)y_2'(x) - y_2(x)y'_1(x) = ke^{-\int p dx},$$

where k is a constant. Since $W(c) = 0$, we see that $k = 0$. Accordingly $W(x_0) = 0$. But $W(x_0) = y_1(x_0)$. Thus we see that y_1 vanishes at x_0 which was any point of (*I*).

3. Equations which involve parameters.† It often happens that the coefficients p, q, r of (1) involve besides the variable x one or more parameters, and it is then important to know the nature of the dependence of the solutions of (1) on these parameters. In what follows we confine our attention to the case of a single parameter, but the methods and results can be immediately extended to the case of any number of parameters.

We consider the differential equation

$$\frac{d^2y}{dx^2} + p(x, \lambda)\frac{dy}{dx} + q(x, \lambda)y = r(x, \lambda), \tag{15}$$

* *Liouville's Journal*, vol. 1 (1836), p. 109.

† Instead of allowing the coefficients of (1) to vary through the introduction of one or more parameters, we may, without any essential complication, adopt the much more general standpoint of the calculus of variations. *Cf.* the paper in the *American Journal* already referred to. The conditions imposed there, at the bottom of p. 315, on the variation of the coefficients might easily be made much less restrictive, as will be seen by comparison with the present paper.

where the real parameter λ is confined to the interval

$$(L) \qquad l_1 \leqq \lambda \leqq l_2.$$

We shall speak of the pair of values (x,λ) as a *point* of the region (I, L) if x is a point of (I) and λ a point of (L). We impose on the coefficients of (15) the following conditions:

(A) *For every value of λ in (L) r is integrable and p and q are absolutely integrable throughout (I). Furthermore the function*

$$\int_c^x r(x,\lambda)dx$$

is a continuous function of the two independent variables (x, λ) at every point of (I, L). Finally, λ_1 being any fixed point and λ_2 a variable point in (L):

$$(16) \qquad \begin{aligned} &\lim_{\lambda_2 = \lambda_1} \int_a^b | p(x, \lambda_2) - p(x, \lambda_1) | \, dx = 0, \\ &\lim_{\lambda_2 = \lambda_1} \int_a^b | q(x, \lambda_2) - q(x, \lambda_1) | \, dx = 0.* \end{aligned}$$

The fundamental theorem is the following where we append an accent to a function of two variables to denote differentiation with regard to the *first* argument.

If $\gamma(\lambda)$ and $\gamma_1(\lambda)$ are continuous functions of λ throughout (L), and c is any point of (I), and the coefficients of (15) satisfy conditions (A), then the solution $y(x, \lambda)$ of (15) determined by the auxiliary conditions

$$y(c, \lambda) = \gamma(\lambda), \quad y'(c, \lambda) = \gamma_1(\lambda)$$

is a continuous function of the two independent variables (x, λ) throughout (I, L); and the same is true of $y'(x, \lambda)$.

This theorem will be proved if we can show that all the terms of the series (4) and (5) are continuous functions of (x, λ) throughout (I, L), and that these series converge uniformly in this region.

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* It can readily be shown that when conditions (16) are fulfilled the integrals

$$\int_c^x p(x, \lambda)dx, \qquad \int_c^x q(x, \lambda)dx$$

are continuous functions of (x, λ). It is an open question whether (16) might not be replaced by the less restrictive conditions that the two integrals last written be continuous functions of (x, λ). *Cf.* foot-note p. 51.

In establishing the continuity of these terms it will evidently be sufficient to consider the functions u_i' since the functions u_i may be written

$$u_0 = \int_c^x u'_0\, dx + \gamma(\lambda),$$

$$u_i = \int_c^x u_i'\, dx \qquad (i = 1, 2, \cdots),$$

so that their continuity as functions of (x, λ) follows from the continuity of the u's.

Since the function u_0' is given by the second formula (3), it is obviously a continuous function of (x, λ) on account of the restriction placed on r in condition (A). Accordingly we use the method of mathematical induction, assuming that it has been proved that u_{n-1} and u'_{n-1} are continuous functions of (x, λ) and then proving that u_n' is continuous. For this purpose we use the value of u_n' given in the second formula (6) which may be written as the negative of the sum of the two integrals

$$(17) \qquad \int_c^x p(\xi, \lambda) u'_{n-1}(\xi, \lambda) d\xi, \qquad \int_c^x q(\xi, \lambda) u_{n-1}(\xi, \lambda) d\xi.$$

We wish to prove that these two integrals are continuous functions of (x, λ). Denoting the first of them by $F(x,\lambda)$ we must then show that

$$\Delta F = F(x + \Delta x, \lambda + \Delta\lambda) - F(x, \lambda)$$

approaches zero as its limit as Δx and $\Delta\lambda$ approach zero independently of each other. Now we have:

$$\Delta F = \int_c^{x+\Delta x} [p(\xi, \lambda + \Delta\lambda) - p(\xi, \lambda)] u'_{n-1}(\xi, \lambda + \Delta\lambda) d\xi$$

$$+ \int_c^{x+\Delta x} p(\xi, \lambda) [u'_{n-1}(\xi, \lambda + \Delta\lambda) - u'_{n-1}(\xi, \lambda)] d\xi,$$

$$+ \int_x^{x+\Delta x} p(\xi, \lambda) u'_{n-1}(\xi, \lambda) d\xi.$$

We will denote these three integrals by ϕ_1, ϕ_2, ϕ_3 respectively, and we will show that they all approach zero as Δx and $\Delta\lambda$ approach zero.

Since u'_{n-1} is continuous, a positive constant K can be chosen so that throughout (I, L)

$$|u'_{n-1}(x, \lambda)| \leqq K.$$

Accordingly we have

$$|\phi_1| \leqq K \int_c^{x+\Delta x} |p(\xi, \lambda+\Delta\lambda) - p(\xi, \lambda)|\,|d\xi|$$
$$\leqq K \int_a^b |p(\xi, \lambda+\Delta\lambda) - p(\xi, \lambda)|\,d\xi,$$

and this is seen to approach zero by (16). Furthermore,

$$|\phi_2| \leqq \int_a^b |p(\xi, \lambda)|\,|u'_{n-1}(\xi, \lambda+\Delta\lambda) - u'_{n-1}(\xi, \lambda)|\,d\xi \leqq \eta \int_a^b |p(\xi, \lambda)|\,d\xi,$$

where η is the greatest value of $u'_{n-1}(x, \lambda+\Delta\lambda) - u'_{n-1}(x, \lambda)$ in (I). Since η approaches the limit zero as $\Delta\lambda$ approaches zero, ϕ_2 does the same.

Finally in considering ϕ_3 let us notice that since p is absolutely integrable throughout (I) and u'_{n-1} is continuous there, pu'_{n-1} is absolutely integrable there, and hence its integral ϕ_3 is a continuous function of the upper limit of integration, *i. e.* of Δx. This shows that ϕ_3 approaches the limit zero with Δx.

We have thus proved that the first of the integrals (17) is a continuous function of (x, λ). The same proof applies to the second if we interchange p and q, and u'_{n-1} and u_{n-1}. The proof that all the terms of (4) and (5) are continuous functions of (x, λ) is thus complete.

To show that these series converge uniformly in (I, L) we have merely to repeat with very slight changes the reasoning given in §2 for establishing their uniform convergence in (I). Thus the constant C must be so chosen that (9) holds throughout (I, L), and this is possible since u_0 and u'_0 are now continuous functions of (x, λ) throughout this region. Furthermore

$$\int_a \Big(|p(\xi, \lambda)| + |q(\xi, \lambda)| \Big)\, d\xi$$

is now a function of λ. It is easily seen by (16) to be continuous throughout (L), and therefore we can introduce a positive constant M so that

$$M \geqq \int_a^b \Big(|p(\xi, \lambda)| + |q(\xi, \lambda)| \Big) d\xi.$$

Formula (11) will then hold and the proof of uniform convergence is complete.

The condition (A) which we have imposed on the coefficients p, q, r is so general that it is worth while for us to notice some simple cases which frequently occur in practice and which are included under it. We mention

here three such cases each of which is included as a special case under the next following one:

(*a*) Condition (*A*) is fulfilled if p, q, r are continuous functions of (x,λ) throughout (I, L).

(*b*) Condition (*A*) is fulfilled if p, q, r are continuous functions of (x, λ) throughout (I, L) except when $x = c_1, c_2, \cdots c_k$; and a positive constant M exists such that except at the last mentioned points $|p| < M$, $|q| < M$, $|r| < M$.

(*c*) Condition (*A*) is fulfilled if except when $x = c_1, c_2, \cdots c_k$

$$p = f(x)\ \phi_1(x,\lambda), \qquad q = f(x)\ \phi_2(x,\lambda), \qquad r = f(x)\ \phi_3(x, \lambda),$$

where ϕ_1, ϕ_2, ϕ_3 are continuous functions of (x,λ) throughout (I, L); and

$$f(x) = (x - c_1)^{a_1}(x - c_2)^{a_2} \cdots (x - c_k)^{a_k},$$

$a_1, \cdots a_k$ being real constants greater than -1.

4. Equations whose coefficients are polynomials in λ. By imposing further restrictions on p, q, r, γ, γ_1 we could make the functions y and y' have certain further properties, for instance that of possessing first partial derivatives with regard to λ which are continuous functions of (x, λ). Without stopping to treat these intermediate possibilities, we will consider what in a sense is an extreme case, and yet one sufficiently general to cover the cases which most frequently occur in practice. The theorem is this:

If c is any point in (I), and γ, γ_1 any constants, and if p, q, r are polynomials in λ whose coefficients are integrable (and in the case of p and q absolutely integrable) functions of x throughout (I), then, $y(x, \lambda)$ being the solution of (15), which satisfies the auxiliary conditions

$$y(c, \lambda) = \gamma, \qquad y'(c, \lambda) = \gamma_1,$$

*$y(x, \lambda)$ and $y'(x, \lambda)$ are continuous functions of (x, λ) when x lies in (I) and λ has any value real or complex. Furthermore, x_0 being any point of (I), $y(x_0, \lambda)$ and $y'(x_0, \lambda)$ are integral analytic functions of λ.**

In order to prove this let us consider a region (Λ) in the complex λ-plane consisting of all the points within and on the boundary of an arbitrarily chosen circle. The functions p, q, r obviously satisfy condition (A) of §3 if in this condition the region (Λ) is substituted for the interval (L). Accordingly we see, from the proof there given, that $y(x, \lambda)$ and $y'(x, \lambda)$ are contin-

* That is, analytic functions having no finite singular points.

uous functions of (x, λ) throughout (I, Λ). Moreover, the series (4) and (5) being uniformly convergent in (I, Λ), will, when we let $x = x_0$, be uniformly convergent in (Λ). From formulæ (3) and (6) we see that the terms of (4) and (5) are polynomials in λ. We see therefore by a fundamental theorem of Weierstrass* that $y(x_0, \lambda)$ and $y'(x_0, \lambda)$ are in (Λ) single valued analytic functions with no singular points. From these facts our theorem follows since the circle (Λ) may be taken as large as we please.

Finally we note that the assumption made in the last theorem that γ and γ_1 are constants is not essential. For if we denote by y_1 and y_2 the solutions of (15) which satisfy the auxiliary conditions:

$$y_1(c, \lambda) = 1, \qquad y_2(c, \lambda) = 0,$$
$$y_1'(c, \lambda) = 0, \qquad y_2'(c, \lambda) = 1,$$

the solution y which satisfies the conditions

$$y(c, \lambda) = \gamma(\lambda), \qquad y'(c, \lambda) = \gamma_1(\lambda)$$

may be written

$$y = \gamma y_1 + \gamma_1 y_2.$$

If then γ and γ_1 are single valued analytic functions with no singular points in a certain continuum (Λ_0) of the λ-plane, we see at once that $y(x, \lambda)$ and $y'(x, \lambda)$ are continuous functions of (x, λ) throughout (I, Λ_0), since by the last theorem this is true of y_1, y_1', y_2, y_2'. For the same reason if x_0 is any point in (I), $y(x_0, \lambda)$ and $y'(x_0, \lambda)$ are single valued analytic functions without singular points in (Λ_0).

5. Roots of functions which involve parameters. Let us consider a real function $\phi(x,\lambda)$ defined for all values of x in (I) when λ belongs to a certain continuous or discontinuous range of values. We suppose that ϕ satisfies the following conditions:

(a) ϕ has a first derivative with regard to x, $\phi'(x, \lambda)$, which is continuous throughout (I) for all values of λ with which we are concerned.

(b) As λ approaches a certain value l (either continuously or in some

* If in a connected two dimensional region of the complex plane a series converges uniformly, and if throughout this region its terms are single valued analytic functions without singular points, then the same will be true of the function represented by the series. *Cf.* for instance Burkhardt, *Analytische Functionen*, 1st edition p. 139.

discontinuous manner as the case may be*), $\phi(x, \lambda)$ and $\phi'(x, \lambda)$ approach uniformly certain limits $f(x)$ and $f'(x)$.

(*c*) The functions $f(x)$ and $f'(x)$ do not both vanish at any point of (I).

(*d*) $f(x)$ vanishes neither at a nor at b.

It should be noticed that it follows from (*a*) and (*b*) that $f(x)$ and $f'(x)$ are continuous throughout (I), and that f' is the derivative of f. From (*c*) it follows that $f(x)$ can not vanish at an infinite number of points in (I), for otherwise these points would have at least one limiting point in (I) at which both f and f' would vanish.

THEOREM. *If $\phi(x, \lambda)$ satisfies conditions (a), (b), (c), (d) and if its limit $f(x)$ has just k roots $x_1, \cdots x_k$ in (I), then for all values of λ sufficiently near to l, $\phi(x, \lambda)$ has just k roots in (I), and these roots approach the values $x_1, \cdots x_k$ as λ approaches l.*

In order to prove this let us suppose the roots $x_1, \cdots x_k$ to be arranged in order of magnitude ($a < x_1 < x_2 < \cdots < x_k < b$), and consider the k intervals

$$(I_i) \qquad |x - x_i| \leqq \delta \qquad (i = 1, 2, \cdots k),$$

where δ is a positive real constant which we suppose taken so small that all the intervals (I_i) lie in (I), and that no two of them have a point in common. Since $f'(x)$ is continuous and, by (*c*), does not vanish at any of the points x_i, δ may be taken so small that $f'(x)$ does not vanish in any of the intervals (I_i). This we suppose done.

We define further the intervals J as follows:

$$(J_0) \qquad a \leqq x \leqq x_1 - \delta,$$

$$(J_i) \qquad x_i + \delta \leqq x \leqq x_{i+1} - \delta \qquad (i = 1, 2, \cdots k - 1),$$

$$(J_k) \qquad x_k + \delta \leqq x \leqq b.$$

Our theorem will be proved if we can show that no matter how small δ may be chosen, a positive ϵ can be taken so small that for every value of λ (in the range with which we are concerned) for which $|\lambda - l| < \epsilon$†, the function $\phi(x,\lambda)$ has no roots in any of the intervals (J_i) and just one root in each of the intervals (I_i). To prove this let us denote by m_1 the smallest value which $|f'(x)|$ has in any of the intervals (I_i), and take ϵ so small that when $|\lambda - l| < \epsilon$

*For instance we might have $l = +\infty$ and the range of values of λ with which we are concerned might be the positive integers.

†If $l = \infty$ we must replace this inequality by $|1/\lambda| < \epsilon$.

the inequality $| \phi'(x, \lambda) - f'(x) | < m_1$ is satisfied. This makes it certain that $\phi'(x, \lambda)$ does not vanish in any of the intervals (I_i) when $|\lambda - l| < \epsilon$. Now denote by m_2 the smallest value of $|f(x)|$ in the intervals (J_i), and take ϵ still smaller than before, if necessary, so that when $|\lambda - l| < \epsilon$ the inequality $|\phi(x,\lambda) - f(x)| < m_2$ is fulfilled; thus making it certain that, for these values of λ, $\phi(x, \lambda)$ does not vanish in any of the intervals (J_i), and has the same sign throughout each of them that $f(x)$ has there. But $f(x)$ changes sign at every root x_i, since $f'(x_i) \neq 0$; accordingly, when $|\lambda - l| < \epsilon$, the sign of $\phi(x, \lambda)$ is alternately plus and minus in the intervals $(J_0), (J_1), \cdots$. Thus we see that $\phi(x, \lambda)$, which has no roots in the intervals (J_i), has at least one root in each of the intervening intervals (I_i). It cannot have more than one root in any interval (I_i), for otherwise $f'(x)$ would vanish there by Rolle's theorem.

We will now apply the theorem which has just been proved to the solutions of linear differential equations of the second order, confining ourselves, however, to the case of homogeneous equations, since here the mere requirement that the solution is not identically zero is sufficient to ensure its satisfying condition (c).

If in the equation

$$\frac{d^2y}{dx^2} + p(x, \lambda)\frac{dy}{dx} + q(x, \lambda)y = 0$$

p and q satisfy conditions (A) of §3, and if $y(x, \lambda)$ is the solution of this equation which satisfies the conditions

$$y(a, \lambda) = \gamma(\lambda), \qquad y'(a, \lambda) = \gamma_1(\lambda),$$

where γ and γ_1 are continuous throughout (L), and if, for all values of λ in (L), $y(x, \lambda)$ has at least k roots in the interval $a < x \leqq b$ which arranged in order of magnitude we denote by $x_1(\lambda), x_2(\lambda), \cdots$, then $x_1, \cdots x_k$ are throughout (L) continuous functions of λ provided one or the other of the following conditions is fulfilled:

(1) *$\gamma(\lambda)$ does not vanish in (L);*

(2) *$\gamma(\lambda) \equiv 0$, and $\gamma_1(\lambda)$ does not vanish in (L).**

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*The object of these last two conditions is to prevent $x_i(\lambda)$ from reaching the value a. This may be attained more generally by the following condition:

(3) γ and γ_1 do not both vanish at any point of (L), and the fraction γ_1/γ has, for values of λ for which $\gamma \neq 0$, if any such exist, a finite lower limit, possibly negative.

The cases mentioned in the text are, however, by far the most important.

We prove this theorem first in case (1). Here we have merely to prove that if l is any value of λ in (L) the functions $x_1, \cdots x_k$ are continuous when $\lambda = l$. Suppose that $y(x, l)$ has just $k + m$ roots in (I). If $y(b, l) \neq 0$ our theorem follows from the one last proved which tells us that when λ is sufficiently near to l, $y(x,\lambda)$ also has exactly $k + m$ roots in (I), and that as λ approaches l, $x_1(\lambda), \cdots x_{k+m}(\lambda)$ approach respectively $x_1(l), \cdots x_{k+m}(l)$. If, however, $y(b, l) = 0$ we must extend the interval (I) a little beyond the point b by defining p and q for values of x a little greater than b. For instance we may let $p(x, \lambda) = q(x,\lambda) = 0$ when $x > b$. If $b' > b$, $y(b', l) \neq 0$ since $y(b, l) = 0$. Accordingly we may apply the first theorem of this section to the interval $a \leqq x \leqq b'$, and we see that, for values of λ sufficiently near to l, $y(x, \lambda)$ has just $k + m$ roots in this interval, and that these roots approach $x_1(l), \cdots x_{k+m}(l)$ as their limits as λ approaches l. Since k of these x's always lie in (I) by hypothesis our theorem is proved.

It remains to prove the theorem when $\gamma(\lambda) \equiv 0$. Here we determine a constant $a' > a$ but differing from a so little that $y(x, \lambda)$ does not vanish in the interval $a < x \leqq a'$ for any value of λ in (L). This we do by taking a' in such a way that $y'(x, \lambda)$ does not vanish in the interval $a \leqq x \leqq a'$, for then y which vanishes when $x = a$ could not vanish again by Rolle's theorem. By applying to the interval $a' \leqq x \leqq b$ the same reasoning which was before applied to (I), the truth of our theorem follows.

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SOME PHYSICAL SOLUTIONS OF THE GENERAL EQUATION OF THE N^{TH} DEGREE

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Successful attempts to solve general equations by physical means date from about 1770. The various devices employed by Plato, Apollonius, Eratosthenes and others in the solution of the Delian problem did indeed afford solutions of the equation $x^3 = 2a^3$, but as is well known the Greeks never rose to a full conception of the general equation of even the third degree.

In 1770 J. Rowning exhibited his Universal Constructor of Equations, constructed on perfectly general principles though the working model was limited to the solution of quadratics. During the century following, L. Lalanne (1840) succeeded in constructing algebraic balances for the solution of equations of as high a degree as the seventh. Since 1870 a great number of devices for the solution of the general equation of the n^{th} degree have been invented, based upon principles of mechanics, optics, hydrodynamics, electricity and magnetism. A brief enumeration of these contrivances and their bibliography is given by R. Mehmke in the *Encyklopädie der Mathematischen Wissenschaften*, vol. I, p. 1067.

The object of the present paper is to present with clearness and in an elementary way several physical solutions which, it seems to the author, deserve to be more widely known, not only because of their intrinsic interest but especially because of their pedagogic value. They offer a remarkable illustration of the inter-relation of the various elementary branches of mathematical science in revealing how various principles may lend themselves to the solution of the same problem. At the same time these solutions exhibit well the unifying effect of the mathematics, in that the most diverse physical principles are seen to conform to the same mathematical laws. Above all they form a good example of the correlation of all science, of how a comparatively difficult problem may find its complete solution in apparently unrelated fields. These

elements, together with the peculiar fascination of the subject, the simplicity of the physical laws involved and the elegance of the underlying mathematical principles, commend these solutions to a place in our elementary text books.

While the ideas underlying the solutions as here presented are not new, the treatment is independent. The detailed description of the various devices has been omitted, for it is needless to say that the interest in these practical devices is essentially theoretical. Mechanical difficulties prevent the actual construction of working models for all but the lowest degrees. The best models as yet constructed give at best but a rough approximation to the roots sought. Besides it may well be questioned, whether since the perfection of approximation methods, such as Horner's, there exists any longer the need for actual solutions by physical means. For this reason the mathematical and physical principles of these devices have been emphasized rather than their mechanical execution.

The first solution is Rowning's,* based upon Segner's† method of evaluating polynomials. The second is due to Lille‡ and has been treated with modifications at some length by Cremona.§ The third solution is based upon Boys'‖ as modified by Grant.¶ The fourth, last in point of discovery, is Meslin's** hydrostatic solution. The fifth solution is that of Lucas,†† though the principle underlying it dates from Gauss.‡‡

1. A kinematic method. It is evident that if a mechanism could be devised for tracing the graph of the polynomial

$$y_n = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n \qquad [1]$$

in rectangular coordinates, the distances from the origin to the points where this graph crosses the x-axis would give at once the real roots of the equation

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0. \qquad [2]$$

* *London Transactions* 1770.
† *Acad. Petropolis Novi Comment.* 1758–59.
‡ *Comptes Rendus*, 1867, p. 854.
§ *Graphical Statics*, Oxford, 1890.
‖ *Philosophic Magazine*, 1886, p. 241.
¶ *American Machinist*, 1896, p. 824.
** *Comptes Rendus*, 1900, p. 888.
†† *Comptes Rendus*, 1890, p. 965.
‡‡ *Werke*, vol. 3, p. 112.

The possibility of such a mechanism is made manifest from the following method of constructing the value of y_n in [1] for any given real value of x.

FIG. 1.

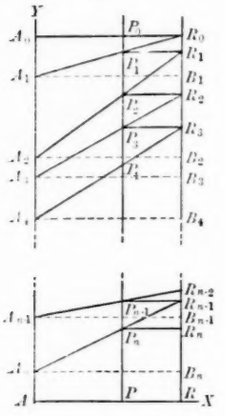


FIG. 2.

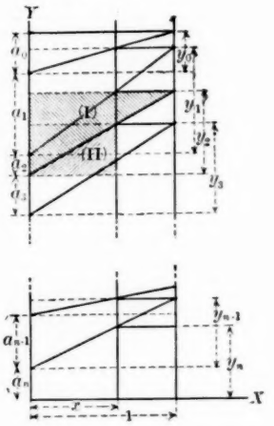
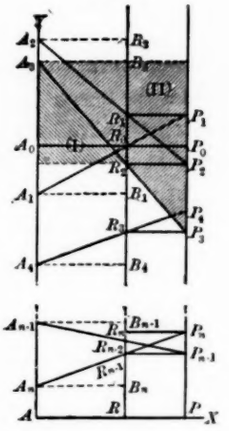


FIG. 3.



Draw the line AX (Figure 1) and on it lay off AR equal to unity and AP equal to the given value of x. Through A draw AY perpendicular to AX, and through P and R draw parallels to AY.

On AY take

$$AA_n = a_n, \quad A_nA_{n-1} = a_{n-1}, \cdots, \quad A_2A_1 = a_1, \quad A_1A_0 = a_0,$$

measuring each distance in the positive or negative direction according as the sign of the corresponding coefficient is positive or negative. Through A_0 draw a line parallel to AX, intersecting the parallels through P and R in P_0 and R_0 respectively.

Draw the line R_0A_1 and let P_1 mark its point of intersection with the line PP_0.

Draw P_1R_1 through P_1 parallel to AX intersecting RR_0 in R_1. Draw R_1A_2 and let its point of intersection with PP_0 be P_2.

Similarly determine in succession the points P_3, P_4, $\cdots$, P_{n-1}, P_n.

The distance PP_n represents the polynomial y_n.

Figure 1 gives the construction for the case in which the coefficients are all positive and $x < 1$. Figure 3 shows the construction when some of the coefficients are negative and $x > 1$.

The proof of the construction follows most readily from Figure 2, which is a repetition of Figure 1, with a more convenient notation. Through the several points $A_1, A_2, A_3, \cdots, A_{n-1}, A_n$, draw lines parallel to AX, intersecting RR_0 in the point $B_1, B_2, B_3, \cdots, B_{n-1}, B_n$ respectively. Moreover let $y_0 = B_1R_0$, $y_1 = B_2R_1$, $\cdots$, $y_{n-1} = B_nR_{n-1}$, $y_n = RR_n = PP_n$.

Now we need only to compare any pair of similar triangles such as those marked (I) and (II) in Figure 2 or 3 to see that for every such pair of triangles there subsists a relation such as

$$y_2 : 1 = (y_3 - a_3) : x$$

from which

$$y_3 = y_2x + a_3.$$

We thus obtain the set of equations

$$\left.\begin{aligned} y_0 &= a_0, \\ y_1 &= y_0x + a_1, \\ y_2 &= y_1x + a_2, \\ &\cdots\cdots\cdots \\ y_{n-1} &= y_{n-2}x + a_{n-1}, \\ y_n &= y_{n-1}x + a_n, \end{aligned}\right\} \qquad [3]$$

from which by successive elimination

$$\begin{aligned} y_1 &= a_0x + a_1, \\ y_2 &= a_0x^2 + a_1x + a_2, \\ y_3 &= a_0x^2 + a_1x^2 + a_2x + a_3, \\ &\cdots\cdots\cdots \\ y_n &= a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n. \end{aligned}$$

This last result may also be obtained otherwise by eliminating the y's in [3] simultaneously as follows. We multiply the first of the equations [3] by x^n, the second by x^{n-1}, and similarly each equation whose first member is y_k by x^{n-k}, and then add the results. The left member of each equation except the last cancels against the first term of the right member of the equation just following and we have remaining

$$y_n = PP_n = a_0x^n + a_2x^{n-1} + a_3x^{n-2} + \cdots + a_{n-1}x + a_n.$$

From the foregoing construction a device for tracing the graph of [1] readily suggests itself. AY, AX and RR_0 are replaced by fixed rods, of which AY carries a graduated scale for the proper determination of the points A_n, A_{n-1}, $\cdots$, A_1, A_0. R_0P_0, R_1P_1, etc., R_0A_1, R_1A_2, etc., and PP_0 are replaced by adjustable rods properly grooved and pivoted so as to admit of parallel motion of the rod PP_0. While this rod PP_0 is made to move parallel to itself a pencil point at P_n traces a graph on a sheet of paper below. The points of intersection of this graph with the line AX furnish the required roots.

2. A solution by visual means. A second solution is suggested by the following method of constructing the polynomial [1].

Let us first assume all the coefficients positive.

FIG. 4.

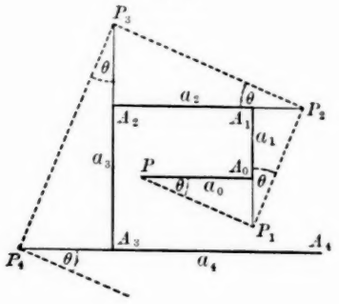
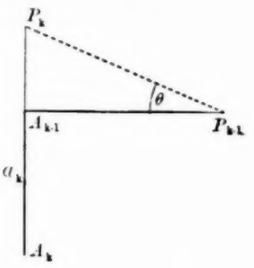


FIG. 5.



Draw $PA_0 = a_0$ (Figure 4), and from A_0 in a direction + 90° with PA_0 draw $A_0A_1 = a_1$.

From A_1 in a direction + 90° with A_0A_1 draw $A_1A_2 = a_2$.

Similarly draw the remaining segments a_3, $\cdots$, a_n of the broken line $PA_0A_1 \cdots A_n$.

Beginning at P draw another rectilinear figure $PP_1P_2 \cdots P_n$ having its vertices P_1, P_2, $\cdots$, P_n respectively on the prolongations of the segments a_1 a_2, $\cdots$, a_n, and the angles at P_1, P_2, $\cdots$, P_{n-1} right angles.

Let angle $P_1PA_0 = \theta$; then each of the angles $P_2P_1A_1$, $P_3P_2A_2$, $\cdots$, $P_nP_{n-1}A_{n-1}$ will also equal θ.

Moreover let

$$PA_0 = y_0 = a_0,\ P_1A_1 = y_1,\ P_2A_2 = y_2,\ \cdots,\ P_nA_n = y_n.$$

Consider now any one of the similar triangles A_0PP_1, $A_1P_1P_2$, $\cdots$, $A_{n-1}P_{n-1}P_n$, say the triangle $A_{k-1}P_{k-1}P_k$ (Figure 5). We have

$$\tan\theta = \frac{P_kA_{k-1}}{P_{k-1}A_{k-1}} = \frac{P_kA_k - A_{k-1}A_k}{P_{k-1}A_{k-1}} = \frac{y_k - a_k}{y_{k-1}},$$

or, putting $\tan\theta = x$,

$$y_k = y_{k-1}x + a_k.$$

Hence, giving to k successively the values $0, 1, 2, \cdots, n$, we obtain equations [3] of §1, for which we have already shown that

$$y_n = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n.$$

In order, therefore, to evaluate this polynomial for a given value of x, say $x = a$, it is only necessary to take $\theta = \tan^{-1}a$, and perform the construction indicated. P_nA_n will represent the corresponding value of the polynomial.

If the coefficients $a_0, a_1, \cdots, a_n$ are not all positive, the construction is modified in that the angle between consecutive segments a_{k+1} and a_k, of the line $PA_0A_1 \cdots A_n$ is taken equal to $-90°$ whenever a_{k+1} and a_k have unlike signs, to $+90°$ when they have like signs.

The reason for this rule lies in the fact that when we change the sign of the middle one of any three segments a_{k-1}, a_k, a_{k+1}, and the sense of the turn which determines its direction, keeping the absolute directions of the segments a_{k-1} and a_{k+1} unchanged, the direction of the segment a_k with reference to each of the segments a_{k-1} and a_{k+1} will have been reversed, so that the change of a single sign is accompanied by two changed turns in the line $PA_0A_1 \cdots A_n$.

Finally the sign of $P_nA_n = y_n$ is determined by the sign of $A_{n-1}A_n = a_n$, and y_n has the same or opposite sign as a_n according as P_nA_n has the same or opposite sense as $A_{n-1}A_n$.

If a is a root of [1], P_nA_n must equal zero. Now $a = \tan\theta$. We shall therefore know the roots of [1], if we can find the values of θ for which P_n coincides with A_n.

These values of θ can be found in various ways. The simplest way is to draw the line $PA_0A_1 \cdots A_n$ on a piece of tracing paper, prolonging the lines A_0A_1, A_1A_2, $\cdots$, $A_{n-1}A_n$ indefinitely in both directions. The tracing is then placed over a sheet of millimeter coordinate paper so that PA_0 will coincide with one of the ruled lines. This line is taken for the line PP_0 and the angle θ is zero. If now we gradually turn the tracing paper in a positive direction

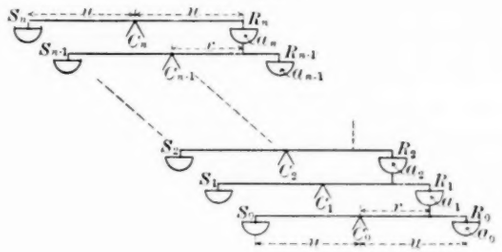
about the point P, increasing the angle θ from zero, it is evident that all positions will be reached for which P_n and A_n coincide. After some experience these positions can readily be determined visually.

3. A dynamic solution. This solution makes use of the principle of moments. Suppose we have a set of balances (Figure 6) :

$$S_0R_0, \quad S_1R_1, \quad S_2R_2, \cdots, S_{n-1}R_{n-1}, \quad S_nR_n,$$

with fulcrums at the middle points C_0, C_1, C_2, $\cdots$, C_{n-1}, C_n respectively. These scale beams are arranged in a vertical plane, successively one above and

FIG. 6.



to the left of the other, so that each right hand scale beam, except the lowest, may be connected at a distance $+ v$ from the fulcrum to the right-hand scale pan of the beam immediately above it. Let the scale beams be of equal length $2u$.

Suppose now that weights

$$a_0, a_1, a_2, \cdots, a_{n-1}, a_n$$

are placed in the scale pans at

$$R_0, R_1, R_2, \cdots, R_{n-1}, R_n$$

respectively, and that y_k denotes the resultant force at any point R_k due to the weight a_k and the weights a_{k-1}, a_{k-2}, $\cdots$, a_1, a_0 in the scale pans of the beams below it.

Then

$$y_{k+1} = X + a_{k+1}, \qquad [4]$$

where X is the force resulting at R_{k+1} from the action of y_k at R . The dis-

tances of R_{k+1} and R_k from C_k are v and u respectively; hence by the principle of moments $vX = y_k u$,

or $$X = y_k x, \qquad [5]$$

where $$x = \frac{u}{v}.$$

Equation [4] therefore becomes

$$y_{k+1} = y_k x + a_{k+1}. \qquad [6]$$

Giving to k successively the values $0, 1, 2, \cdots, n-2, n-1$, and remembering that $y_0 = a_0$, we obtain once more the equations [3] of §1, from which equation [1] follows as before.

If the weight a_{k+1} had been placed in the left pan, at S_{k+1}, instead of in the right pan at R_{k+1}, equation [4] would obviously be replaced by

$$y_{k+1} = X - a_{k+1},$$

so that for every weight placed in a left pan, the corresponding coefficient in equations [3], and consequently also in equation [1], is negative.

Now since y_n denotes the resultant force effective at R_n, the following dynamic method of evaluating any polynomial of the form [1], for any real value of x, suggests itself.

We adjust our system of scale beams, n in number, by varying v so that $\frac{u}{v} = x$. We then place in one of the pans of each of the beams S_0R_0, S_1R_1, $\cdots$, S_nR_n weights respectively proportional to the coefficients $a_0, a_1, \cdots, a_n$ of the polynomial, in the right or left pan according as the corresponding coefficient is positive or negative. The counterweight required to balance the beam S_nR_n represents the value of the polynomial for the given value of x. y_n is positive or negative according as the counterweight must be placed at S_n or at R_n in order to balance the system.

To determine the roots of the equation [2], the scale beams must be connected so that the system may be balanced by changing the distance v instead of by the use of counterweights. This is accomplished by attaching the fulcrums $C_1, C_2, \cdots, C_n$ to parallel slides, each moved by a separate screw. These screws are geared together in such a way that as C_1 moves through a certain distance, C_2 moves through twice that distance, C_n through n times that distance. At the same time each beam slides through the ring by which the scale pan at the R above it is attached to it. Thus however the distance v may be altered, it will have the same value for each beam.

The pans having been properly weighted, the distance v is gradually varied until the system of scales is in equilibrium. For each v that this occurs the equation

$$x = \frac{u}{v}$$

furnishes a root of the equation [2].

Theoretically v may have any real value negative as well as positive. But for practical reasons v is positive and at most equal to unity, hence x is necessarily positive and not less than unity. But since the roots of

$$a_0x^n - a_1x^{n-1} + a_2x^{n-2} - \cdots \pm a_{n-1}\,x \mp a_n = 0 \qquad [7]$$

are the same as those of [2] with the signs changed, and those of

$$a_nx^n + a_{n-1}\,x^{n-1} + a_{n-2}\,x^{n-2} + \cdots + a_1x + a_0 = 0 \qquad [8]$$

the reciprocals of the roots of equation [2], the positive roots of [7] with the sign changed will give the negative roots of [2] and the reciprocals of the roots of [8] whose absolute values are less than unity, will yield those roots of [2] whose absolute values are greater than unity.

4. A solution by hydrodynamics. In the preceding solution the balance was brought into equilibrium by changing the lengths of the lever-arms at which the weights act. Obviously equilibrium might also be established without changing the distances by a gradual variation of the weights.

Referring again to the polynomial [1], suppose that at the distances $a_0, a_1, a_n, \cdots, a_{n-1}$ from the fulcrum C of a scale beam (Figure 7), to the left or right of C according as the coefficient is positive or negative, the respective weights

$$W_n, W_{n-1}, W_{n-2}, \cdots, W_1$$

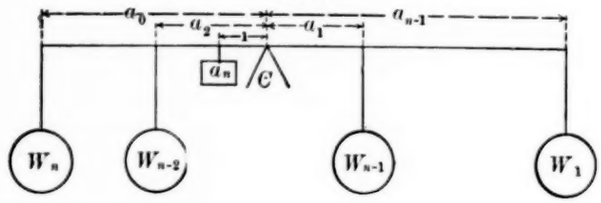
be attached, and at a distance unity from the fulcrum a weight a_n, to the left or right according as a_n is positive or negative. The moment of rotation y_n of the system in a positive direction about C will then be

$$a_0\,W_n + a_1\,W_{n-1} + a_2\,W_{n-2} + \cdots + a_{n-1}\,W_1 + a_n. \qquad [9]$$

Suppose now that each weight W_k consists of a vessel containing water

and that the shape of the vessel is such that the volume, and consequently the weight of its contents, varies as the k^{th} power of the height to which it is filled.

FIG. 7.



Disregarding the weights of the empty vessels and supposing that they are all filled to a common height x we should have

$$W_n = cx^n, W_{n-1} = cx^{n-1}, \cdots, W_2 = cx^2, W_1 = cx, \qquad [10]$$

where c is an arbitrary factor of proportionality which may be so chosen that the height of the water in the vessels is expressed in the same unit as the distances from c at which the weights are attached. If the weights are in grams and the distances in centimeters, c must be put equal to unity, since the weight of a given quantity of water in grams is equal to its volume in cubic centimeters. With $c = 1$ the values [10], substituted in [9], give us the polynomial [1].

If therefore we could construct vessels having the assumed forms, the value of the polynomial [1], for a positive value x of the variable, might be found from the counterweight necessary to balance the system after the vessels have been filled with water to a common height x. If the units used are grams and centimeters, the counterweight at the unit distance will give y_n directly, and it is positive or negative according as the counterweight is required at the right or the left of the fulcrum.

Now it is easy to show that vessels of the assumed form can actually be constructed. In fact for every value of k there exists a solid of revolution which satisfies the required condition that its volume equals the k^{th} power of its altitude.

For supposing that such a solid of revolution exists, take its axis for the x-axis, a tangent at its vertex for the y-axis; and let $y = f(x)$ represent the equation of a meridian of the solid. Putting V for the volume cut off from

the solid by a plane perpendicular to the x-axis at a distance x, we have on the one hand, by the principles of the calculus,

$$V = \int_0^x \pi y^2 dx,$$

and on the other hand, by our assumption regarding the form of the vessel,

$$V = x^k;$$

whence

$$x^k = \int_0^x \pi y^2 dx.$$

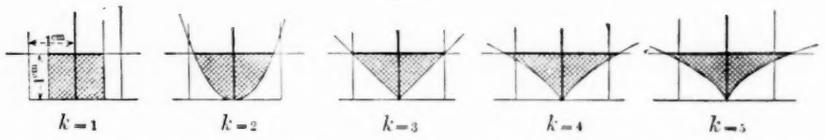
Differentiating this last equation with respect to x we obtain $kx^{k-1} = \pi y^2$, or

$$y = \sqrt{\frac{k}{\pi}} \cdot x^{\frac{k-1}{2}}. \qquad [11]$$

For $k = 1, 2, 3, 4, 5$, we have respectively $y = 0.564, (0.798)x^{\frac{1}{2}}, (0.977)x, (1.128)x^{\frac{2}{3}}, (1.262)x^{\frac{2}{3}}$.

The meridian sections of the corresponding vessels are shown in Figure 8.

FIG. 8.



The values of x for which the moment of rotation y_n is zero, that is, for which the system is in equilibrium without the use of counterweights, give the roots of the equation [2].

These values could be found by gradually increasing the height of the water in the vessels, keeping it at the same height in all the vessels, and observing the successive heights for which the system balances. The difficulty of keeping the height exactly the same in all the vessels is overcome by replacing the vessels by heavy solids of revolution of the same shapes and immersing these in water to a common depth. By the principle of Archimedes the upward force of the displaced water is exactly equal to the former downward force of the filled vessels. But the direction of the forces having been reversed, the weight a_n must now be attached at the opposite side of the fulcrum, that is at a unit distance to the right of C if the coefficient a_n is positive, to the left

if negative. The gradual immersion of all the bodies to an equal depth is easily effected by placing the apparatus in a vessel into which water may be slowly admitted. The bodies are adjusted so that their lowest points lie in the same horizontal plane. The effect of the weights of the bodies is eliminated by carefully balancing the system before attaching the weight a_n. Then the weight a_n is attached and water admitted to the vessel. The successive depths of immersion at which equilibrium takes place give the positive roots of equation [2]. To find the negative roots we first transform the equation into another whose roots are equal to the roots of the given equation with contrary signs. The positive roots of the transformed equation will, of course, give the negative roots of the original equation.

5. An electromagnetic solution. The preceding methods have dealt with real roots of equations only; the present method furnishes all the roots of an equation. We shall therefore write the polynomial in the form

$$f(z) = a_0 z^n + a_1 z^{n-1} + \cdots a_{n-1} z + a_n \qquad [12]$$

and seek the values of

$$z = x + iy$$

which will render $f(z) = 0$.

Take $n + 1$ arbitrary constants $\lambda_1, \lambda_2, \cdots, \lambda_n, \lambda_{n+1}$, form the function

$$F(z) = (z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_n)(z - \lambda_{n+1}),$$

and decompose the rational fraction $f(z)/F(z)$ into partial fractions. We get

$$\frac{f(z)}{F(z)} = \frac{M_1}{z - \lambda_1} + \frac{M_2}{z - \lambda_2} + \cdots + \frac{M_n}{z - \lambda_n} + \frac{M_{n+1}}{z - \lambda_{n+1}}, \qquad [13]$$

where

$$M_1, M_2, \cdots, M_n, M_{n+1}$$

are constants given by the relation

$$M_k = \frac{f(\lambda_k)}{F'(\lambda_k)},$$

$F'(z)$ denoting as usual the derivative of $F(z)$ with respect to z.

We next take some non-conductor of electricity, say a pane of glass or a sheet of paper, to represent a plane of complex numbers; on it we select a convenient line as an x-axis, and along this line, from a point O as origin,

measure the distances OR_1, OR_2, $\cdots$, OR_n, OR_{n+1} equal respectively to λ_1, λ_2, $\cdots$, λ_n, λ_{n+1}. At the points R_1, R_2, $\cdots$, R_n, R_{n+1} let the horizontal plane or pane be traversed at right angles by electric currents whose strengths are proportioned to M_1, M_2, $\cdots$, M_n, M_{n+1} respectively, with their directions downward for positive M's, upward for negative M's.

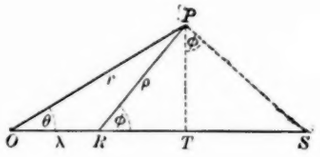
The relative current strengths are easily obtained by dividing a single circuit into branches by means of wires of uniform thickness whose various lengths are inversely proportional to the M's. The strengths of the currents carried by these branches vary inversely as the resistances (Ohm's Law), the resistances vary directly as the lengths of the wires, and the lengths of the wires vary inversely as the quantities M; consequently the current-strengths vary directly as the quantities M.

Let

$$z = x + iy$$

represent any point P in our pane (Figure 9).

FIG. 9.



By Biot and Savart's law the force exerted upon a small magnet at P by a current of indefinite length and intensity M, descending perpendicularly at R, is directly proportional to M and inversely to ρ, the distance between R and P. Its direction is PS, the normal to RP. Hence denoting this force by F, we may write

$$F = \frac{kM}{\rho}, \qquad [14]$$

where k is the factor of proportionality.

Let ϕ denote the angle which the line RP makes with the x-axis; then the angle which PS makes with PT, the perpendicular from P to the x-axis, is also ϕ. The components X, Y of the form F in the positive directions of the coordinate axes are therefore respectively

$$X = F \sin \phi, \qquad Y = -F \cos \phi,$$

whence

$$X = \frac{kMy}{\rho^2}, \qquad Y = \frac{-kM(x-\lambda)}{\rho^2}, \qquad [15]$$

since in the triangle RPT

$$\sin\phi = \frac{TP}{RP} = \frac{y}{\rho}, \qquad \cos\phi = \frac{RT}{RP} = \frac{x-\lambda}{\rho}.$$

Each of the currents at $R_1, R_2, \cdots, R_n, R_{n+1}$ gives rise to x and y components of the form [15]. Hence the resultant force at P in the direction of the x-axis is given by the sum

$$\Sigma \frac{kMy}{\rho^2} = k\Sigma \frac{My}{\rho^2} = k\left[\frac{M_1 y}{\rho_1^2} + \frac{M_2 y}{\rho_2} + \cdots + \frac{M_n y}{\rho_n} + \frac{M_{n+1}\, y}{\rho_{n+1}}\right],$$

and the resultant in the direction of the y-axis by

$$\Sigma \frac{-kM(x-\lambda)}{\rho^2} = -k\,\Sigma \frac{M(x-\lambda)}{\rho^2}$$

$$= -k\left[\frac{M_1(x-\lambda_1)}{\rho_1^2} + \frac{M_2(x-\lambda_2)}{\rho_2^2} + \cdots + \frac{M_n(x-\lambda_n)}{\rho_n^2} + \frac{M_{n+1}(x-\lambda_{n+1})}{\rho_{n+1}^2}\right].$$

At the neutral points of the electric field we have

$$\Sigma \frac{My}{\rho^2} = 0, \qquad \Sigma \frac{M(x-\lambda)}{\rho^2} = 0, \qquad (16)$$

and therefore also

$$\Sigma \frac{M(x-\lambda)}{\rho^2} - i\,\Sigma \frac{My}{\rho^2} = \Sigma\left(M \frac{x-\lambda-iy}{\rho^2}\right) = 0.$$

But

$$\rho^2 = (x-\lambda)^2 + y^2 = (x-\lambda+iy)(x-\lambda-iy),$$

whence

$$\Sigma\left(M \frac{x-\lambda-iy}{\rho^2}\right) = \Sigma \frac{M}{x-\lambda+iy} = \Sigma \frac{M}{z-\lambda} = \frac{f(z)}{F(z)} = 0$$

by [13], and consequently also

$$f(z) = 0.$$

It appears, therefore, that the coordinates of every neutral point in our plane give a value of z for which $f(z) = 0$. Moreover for every root of

$f(z) = 0$ there exists a neutral point in our field, since every step that led from [16] to $f(z) = 0$ can be reversed.

The neutral points in question can be determined by various methods. The most obvious method is to scatter iron filings over the horizontal pane which represents the plane of complex numbers. By slightly tapping the pane while the current is flowing, the filings arrange themselves in lines, representing the lines of force. The neutral points or points of equilibrium can then be easily distinguished.

If the non-conducting pane is replaced by a metallic plate which receives the currents for which M is positive and discharges them when M is negative, then the neutral points which we have considered are approximately replaced (the more accurately the larger the plate) by the nodal points of the system of equipotential lines and as such may be located either electrolytically or by means of the galvanometer.

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THE BALLISTIC PROBLEM

By Frank Gilman

The problem of the motion of a projectile in a medium resisting as any power of the velocity was first solved by John Bernouilli in 1718, but he expressed the coordinates of any point of the trajectory in a form where a final integration is required to determine them numerically; and, as far as the general problem is concerned, no later mathematician has improved on his results.

Modern experiments in gunnery, notably those conducted for the British government (1865–1880) by General Francis Bashforth, have established the fact that for moderate velocities, *i. e.* from 100 to 900 feet a second, the resistance of the air to the passage of the projectile is very nearly proportional to the square of the velocity; that for greater velocities *i. e.* from 900 to 1300 feet a second it is nearly proportional to the cube of the velocity; and that for high velocities, *i. e.* from 1300 to 2780 feet a second it is again very closely proportional to the square of the velocity.

In 1873 General Bashforth published a report of his experiments accompanied with the mathematical theory for the case where the resistance of the air is as the cube of the velocity, and an elaborate set of tables based on the method of quadratures for the integration of the final equations.

In 1890 he published a revised report in which the more difficult case where the resistance is as the square of the velocity was treated in like manner.

The same subject has been somewhat differently treated by Niven,* Siacci,† and Ingalls.‡

In this paper I shall attempt to show that by means of the application of the method of Least Squares to the development of functions the differential equations for the trajectory when the resistance is as the square of the velocity can be integrated without excessive labor, so that they will give results of very considerable exactness.

I shall first consider a method of very great accuracy but requiring much

* Niven, *Proceedings of the Royal Society*, London, Vol. 26, 1877.

† Siacci, *Balistique Extérieure;* Traduction annotée par P. Laurent; Paris, 1892.

‡ Ingalls, *Exterior Ballistics*, Washington, 1900.

labor in its application, and I shall afterwards develop much simpler working methods of sufficient accuracy for practical purposes.

The following* are the well known differential equations of motion for the case under consideration:

$$cdx = \frac{dp}{p\sqrt{1+p^2} + \log(p+\sqrt{1+p^2}) - C}, \tag{1}$$

$$cdy = \frac{pdp}{p\sqrt{1+p^2} + \log(p+\sqrt{1+p^2}) - C}, \tag{2}$$

$$\sqrt{cg}.dt = \frac{-dp}{\sqrt{C - p\sqrt{1+p^2} - \log(p+\sqrt{1+p^2})}}, \tag{3}$$

in which dx denotes the differential of the abscissa and dy the differential of the ordinate, while p, or $\frac{dy}{dx}$, is the tangent of the angle at which the trajectory is inclined to the horizontal.

We have also

$$C = \frac{1}{2ch\cos^2\phi_0} + p_0\sqrt{1+p_0^2} + \log(p_0+\sqrt{1+p_0^2})$$

in which $c = \frac{g}{k^2}$, g being the acceleration of gravity and k the velocity of the projectile when the resistance becomes equal to its weight; $h = \frac{v_0^2}{2g}$ where v_0 is the initial velocity; ϕ_0 is the initial angle of elevation and p_0 the initial value of p. It is easily seen that C is positive and always greater than

$$p\sqrt{1+p^2} + \log(p+\sqrt{1+p^2}.)$$

First method. Let us replace the transcendental function $p\sqrt{1+p^2} + \log(p+\sqrt{1+p^2})$ which we will represent by $f(p)$ by the rational expression $a'p + b'p^3 + c'p^5$ in which the coefficients a', b', and c', are to be calculated by the method of Least Squares, the general form of the equation of condition being as follows:

$$a'p + b'p^3 + c'p^5 - f(p) = 0.$$

Make successively in this equation $p = 0.1$, $p = 0.2$, etc. up to $p = 1.0$,

* v. Bartlett's *Analytical Mechanics.*

thus forming 10 equations of condition. Forming the normal equations and solving them we obtain

$$\begin{aligned} \text{Log } a' &= 0.3012124 \\ \text{Log } b' &= 9.5126277 \\ \text{Log } (-c') &= 8.4899010 \end{aligned}$$

A more detailed exposition of this method of developing irrational and transcendental expressions is given in an article that was published in the *Journal of The Franklin Institute* for June, 1894, entitled "Application of the Method of Least Squares to the Development of Functions."

The above values of a', b', and c' when substituted in the rational expression will make it give values for $f(p)$ whose greatest error will hardly exceed one in the fourth place of decimals, while the integrals, cx and cy, are likely to be even more exact.

In formulas developed by the method of Least Squares (especially when the number of conditional equations is large) the sum of the positive errors in a

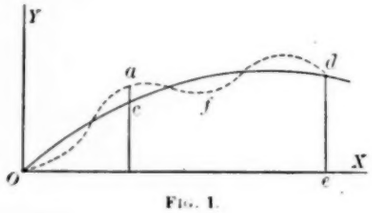

FIG. 1.

series of results, will be nearly equal to the sum of the negative errors Therefore if the line *Oafd* represents the course of a given curve with reference to the axes x and y, its location as determined by the method of Least Squares is shown by the curve *Ocd*; and although these curves may deviate sensibly from each other at certain points, as at a and c, yet the areas included between the lines *Oe*, *ed* and each curve will be practically the same. In other words the corresponding definite integrals will be nearly equal.

The differential equations (1) and (2) may now be written as follows:

$$cc'dx = \frac{dp}{p^5 + \frac{b'}{c'}p^3 + \frac{a'}{c'}p - \frac{C}{c'}}, \tag{4}$$

$$cc'dy = \frac{pdp}{p^5 + \frac{b'}{c'}p^3 + \frac{a'}{c'}p - \frac{C}{c'}}. \tag{5}$$

These equations can now be integrated by the method of decomposition into partial fractions.

Let $l = \frac{b'}{c'}$, $m = \frac{a'}{c'}$, $n = -\frac{C}{c'}$ and observe that l and m are negative and n positive.

By Descartes' Rule of Signs $p^5 + lp^3 + mp + n = 0$ has two positive real roots, one negative real root and two imaginary roots; call them α, β, γ, $\delta + \epsilon\sqrt{-1}$, $\delta - \epsilon\sqrt{-1}$. Then

$$\frac{1}{p^5 + lp^3 + mp + n} \equiv \frac{L}{p-\alpha} + \frac{M}{p-\beta} + \frac{N}{p-\gamma} + \frac{Pp+Q}{(p-\delta)^2+\epsilon^2}, \tag{6}$$

$$L = \left[\frac{p-\alpha}{p^5 + lp^3 + mp + n}\right]_{p=\alpha} = \frac{1}{5\alpha^4 + 3l\alpha^2 + m}.$$

Similarly,

$$M = \frac{1}{5\beta^4 + 3l\beta^2 + m},$$

$$N = \frac{1}{5\gamma^4 + 3l\gamma^2 + m}.$$

Clearing (6) of fractions and writing the coefficients of p^4 and p^3 equal to zero, noting that $\alpha + \beta + \gamma + 2\delta = 0$, we have

$$L + M + N + P = 0, \text{ and } L\alpha + M\beta + N\gamma + 2P\delta + Q = 0,$$

whence

$$P = -(L + M + N),$$

$$Q = -(L\alpha + M\beta + N\gamma + 2P\delta).$$

Equations (4) and (5) now integrate into

$$cc'x = L\log(p-\alpha) + M\log(p-\beta) + N\log(p-\gamma) + \frac{P}{2}\log[(p-\delta)^2+\epsilon^2] + \frac{P\delta+Q}{\epsilon}\tan^{-1}\left(\frac{p-\delta}{\epsilon}\right), \tag{7}$$

$$cc'y = L\alpha\log(p-\alpha) + M\beta\log(p-\beta) + N\gamma\log(p-\gamma) + \frac{2P\delta+Q}{2}\log[(p-\delta)^2+\epsilon^2] + \frac{\delta(P\delta+Q) - P\epsilon^2}{\epsilon}\tan^{-1}\left(\frac{p-\delta}{\epsilon}\right). \tag{8}$$

The constants of integration are determined from the condition that when p is equal to its initial value p_0, $x = 0$ and $y = 0$; but the formulas may be used

as they stand without the determination of the constants of integration if p is taken between proper limits.

We will now apply these equations in solving the following numerical example which is given by Professor Bashforth in his treatise on the Chronograph, page 97.

Initial angle of elevation $\phi_0 = 30°$; initial velocity $V = 751$ ft. per sec.; diameter of shot $= 6.27$ in.; weight $= 70$lbs. Here $c = 0.0000339822$, $C = 3.455425$.

We first find the five roots of the equation

$$p^5 + lp^3 + mp + n = 0,$$

in which $l = \frac{b'}{c'} = -10.5372354$, $m = \frac{a'}{c'} = -64.7606866$, $n = -\frac{C}{c'} = +111.84085$.

The following are the values of the roots:

$$\alpha = +1.3779718$$
$$\beta = +3.6221343$$
$$\gamma = -4.0293468$$
$$\delta = -\frac{\alpha + \beta + \gamma}{2} = -0.4853796$$
$$\epsilon^2 = 3\delta^2 + l - \gamma(\alpha + \beta) - \alpha\beta = +5.3255100$$

Next computing the five constants by the formulas of the preceding page we get:

$$L = -0.0093670, \quad M = +0.0026236, \quad N = +0.0013514,$$
$$P = +0.0053920, \quad Q = +0.0140848.$$

It is evident that the accuracy of the integrals as given by equations (7) and (8) will depend on the exactness with which the following equation is satisfied:

$$\frac{1}{p\sqrt{1+p^2} + \log(p + \sqrt{1+p^2}) - C} = \frac{1}{c'}\left[\frac{L}{p-\alpha} + \frac{M}{p-\beta} + \frac{N}{p-\gamma} + \frac{Pp+Q}{(p-\delta)^2 + \epsilon^2}\right]. \quad (9)$$

Substituting in this equation the values of the roots and constants already

found, we obtain the following table, which gives the values of the two members of this equation corresponding to different values of p:

p	FIRST MEMBER	SECOND MEMBER
0.05	0.29803	0.29803
0.10	0.30721	0.30722
0.15	0.31703	0.31704
0.20	0.32757	0.32758
0.25	0.33895	0.33896
0.30	0.35130	0.35132
0.35	0.36478	0.36479
0.40	0.37957	0.37957
0.45	0.39590	0.39589
0.50	0.41404	0.41403
0.55	0.43436	0.43434
0.60	0.45727	0.45725
0.65	0.48334	0.48331
0.70	0.51327	0.51325
0.75	0.54801	0.54801

In accordance with what has been said in connection with figure 1, it follows that the integrals are likely to be even more exact than might be inferred from the indications of the above table.

Substituting the values of the roots and constants in equation (7), and taking it between the limits $p = \tan 30°$ and $p = 0$, we find $x = 6026.8$ feet, which is the horizontal distance from the origin to where the projectile attains its greatest elevation.

Professor Bashforth finds by quadratures $x = 6032$ ft., which is 5 feet in error.

Taking equation (8) between the same limits, we obtain $y = 1864.4$ ft. Professor Bashforth finds $y = 1867$ ft. In order to find the range we will assume with Bashforth that the angle of fall $= -(37° 3')$, and taking equations (7) and (8) between the limits $p = \tan 30°$ and $p = \tan -(37° 3')$, we obtain

$$x = 11326.1 \text{ ft.} = \text{range},$$
$$y = -\ 7.2 \text{ ft.}$$

According to Bashforth, $x = 11334$ ft., and $y = -6.5$ ft., showing an error of about 8 feet in the range.

The equations (7) and (9) undoubtedly give the coordinates of the trajectory correct to the nearest foot; but their use involves great labor, and they are chiefly valuable as a criterion by which to compare the results of simpler methods.

Second method. If instead of replacing our $f(p) \equiv p\sqrt{1+p^2} + \log(p + \sqrt{1+p^2})$ by $a'p + b'p^3 + c'p^5$ we assume $f(p) = ap + bp^2$, a and b being constants to be determined as before, our method while open to the objection that two separate sets of formulas are required one for the ascending and the other for the descending portion of the trajectory, is very much less laborious in practice than the one we have been considering.

Since $f(-p) = -f(p)$, if we determine a and b on the assumption that p is positive we must use $ap - bp^2$ for $f(p)$ when p is negative.

Integrating the differential equations (1), (2), and (3) on the assumption that $f(p) = ap + bp^2$ we get for the ascending portion of the trajectory:

$$x = \frac{1}{c\sqrt{a^2+4bC}} \log \frac{a + 2bp - \sqrt{a^2+4bC}}{a + 2bp + \sqrt{a^2+4bC}}, \tag{10}$$

$$y = \frac{1}{2bc} \log(C - ap - bp^2) - \frac{ax}{2b}, \tag{11}$$

$$t = \frac{1}{\sqrt{bcg}} \sin^{-1}\left(\frac{a+2bp}{\sqrt{a^2+4bC}}\right). \tag{12}$$

For the descending portion where $f(p) = ap - bp^2$ we get:

$$x = \frac{1}{c\sqrt{a^2-4bC}} \log \frac{a - 2bp - \sqrt{a^2-4bC}}{a - 2bp + \sqrt{a^2-4bC}}, \text{ if } a^2 > 4bC, \tag{13}$$

$$x = \frac{2}{c\sqrt{4bC-a^2}} \tan^{-1}\left(\frac{a-2bp}{\sqrt{4bC-a^2}}\right), \qquad \text{if } a^2 < 4bC, \tag{13'}$$

$$y = -\frac{1}{2bc} \log(C - ap + bp^2) + \frac{ax}{2b}, \tag{14}$$

$$t = \frac{1}{\sqrt{bcg}} \log\left[\sqrt{C - ap + bp^2} - p\sqrt{b} + \frac{a}{2\sqrt{b}}\right]. \tag{15}$$

As the constants of integration have not been introduced in the above formulas, we must remember in using them to take p between the limits for which x, y and t are to be calculated.

The following table gives the values of a and b corresponding to different angles of elevation.

p	ϕ	$\log a$	$\log b$	a	a^2
0.1	5° 43′	0.3007161	8.6654758	1.998555	3.994222
0.2	11 19	0.2998279	8.9559684	1.994472	3.977917
0.3	16 42	0.2983910	9.1254675	1.987884	3.951682
0.4	21 48	0.2964379	9.2440980	1.978964	3.916298
0.5	26 34	0.2940089	9.3342507	1.967927	3.872735
0.6	30 58	0.2911508	9.4060248	1.955018	3.822097
0.7	35 0	0.2879118	9.4649223	1.940492	3.765508
0.8	38 40	0.2843400	9.5142827	1.924598	3.704077
0.9	41 59	0.2804821	9.5563036	1.907577	3.638850
1.0	45 0	0.2763813	9.5925171	1.889650	3.570776

In using this table no interpolations will be necessary, as either one of two consecutive sets of values of the a and b functions can be used without making any material difference in the results. The quantities a and b in the table are determined in the usual manner by the method of Least Squares, the general form of the equation of condition being

$$ap + bp^2 - f(p) = 0.$$

Make successively in this equation $p = 0.01$, $p = 0.02$, etc., up to $p = 0.10$, thus forming 10 equations of condition. Solving the resulting normal equations, we obtain the $\log a$ and $\log b$ given in the first horizontal line of the table. In the same manner the $\log a$ and $\log b$ found in the second horizontal line of the table are formed from 20 equations of condition, and those in the last horizontal line are formed from 100 equations of condition.

We will now solve by the above equations and table the numerical example that we have already worked by the first method, and in which $C = 3.455425$, $c = 0.0000339822$. Since in this example the initial angle of elevation equals 30°, we take the values of a and b from that line in the table corresponding to $p = 0.6$. Substituting these quantities together with the values of C and c in equations (10) (11) and (12) and taking them between the limits corresponding to $p = \tan 30°$ and $p = 0$, we obtain

$$x = 6026.7 \text{ ft.},$$
$$y = 1864.8 \text{ ft.},$$
$$t = 10.38 \text{ sec.}$$

As the angle of fall is 37° 3′, we take for the descending branch the values of a and b corresponding to $p = 0.8$. Substituting these in the equations (13′) (14) and (15) and taking them between the limits $p = \tan(37° 3')$ and $p = 0$, we find

$$x = \quad 5300.7 \text{ ft.},$$
$$y = -\ 1871.1 \text{ ft.},$$
$$t = \quad 11.13 \text{ sec.}$$

Range = 6026.7 + 5300.7 = 11327.4 ft.

Time of flight = 10. 38 + 11.13 = 21.51 sec.

If we use the values of a and b corresponding to $p = 0.8$ for the ascending branch of the trajectory, we obtain the following for the x, y and t of this branch:

$$x = 6026.\ 2 \text{ ft.},$$
$$y = 1865.\ 3 \text{ ft.},$$
$$t = \quad 10.\ 38 \text{ sec.}$$

From this it appears that we may generally use the same values of a and b for the ascending as for the descending branch, those quantities being taken which correspond to the angle of fall.

We will illustrate by one more example taken from Capt. Ingalls' *Treatise on Exterior Ballistics*, page 104.

A 12-inch service projectile is fired at an angle of departure of 10°, with an initial velocity of 1886 ft. per sec.; it is required to find x_1, y_1, t_1, corresponding to $p = 0$, and x_2, y_2, t_2 for the point where the angle of fall is $-13°$, both sets of coordinates being referred to the mouth of the gun.

In this example $C = 0.72037$, $c = 0.0000254985$. Take a and b from that line of the table corresponding $p = 0.\ 2$. Substituting in equations (10), (11) and (12), and taking p between the limits tan 10° and 0, we find

$$x_1 = 13203.8 \text{ ft.},$$
$$y_1 = \quad 1294.4 \text{ ft.},$$
$$t_1 = 8.46 \text{ sec.}$$

For the descending branch take the a and b corresponding to $p = 0.3$, and substitute in equations (13) (14) and (15), taking p between the limits

tan $(-13°)$ and 0. We thus obtain the following values of the coordinates referred to the vertex of the trajectory:

$$x = 9705.3 \text{ ft.},$$
$$y = -1027.3 \text{ ft.},$$
$$t = 8.32 \text{ sec.}$$

Whence

$$x_2 = 13203.8 + 9705.3 = 22909.1 \text{ ft.},$$
$$y_2 = 1294.4 - 1027.3 = 267.1 \text{ ft.},$$
$$t_2 = 8.46 + 8.32 = 16.78 \text{ sec.}$$

Capt. Ingalls working by "Niven's Method" obtains the following values for these quantities:

$$x_1 = 13180.7 \text{ ft.},$$
$$y_1 = 1291.8 \text{ ft.},$$
$$x_2 = 23007.0 \text{ ft.},$$
$$y_2 = 241.7 \text{ ft.},$$
$$t_1 = 8.44 \text{ sec.},$$
$$t_2 = 16.83 \text{ sec.}$$

If the actual law of the resistance is that of the square of the velocity these results of Capt. Ingalls are somewhat erroneous, the x_2 being 98 feet in error.

The more exact equations (7) and (8) give the following values for these coordinates:

$$x_1 = 13205.6 \text{ ft.},$$
$$y_1 = 1295.5 \text{ ft.},$$
$$x_2 = 22908.9 \text{ ft.},$$
$$y_2 = 267.3 \text{ ft.}$$

Third method. A rougher method which in point of simplicity leaves little to be desired, and yet is surprisingly accurate in application is based on the assumption that $f(p) \equiv p\sqrt{1+p^2} + \log(p + \sqrt{1+p^2}) = ap$.

Equations (1), (2) and (3) integrate at once into

$$cx = \frac{1}{a}\log(C - ap), \tag{16}$$

$$cy = \frac{1}{a}\left[p + \frac{C}{a}\log(C - ap)\right], \tag{17}$$

$$\sqrt{cg}\cdot t = \frac{2}{a}\sqrt{C - ap}, \tag{18}$$

and the ascending and descending portions of the trajectory do not need to be dealt with separately.

Here again as the constants of integration are not written we must in using the equations take the difference between the values corresponding to the extreme values of p.

The value of a is to be determined by the method of Least Squares, our equation of condition being $ap - f(p) = 0$; but as we have but one unknown we have merely to compute the arithmetical mean of the values of $\frac{f(p)}{p}$.

The following table gives the values of a for different angles of elevation:

p	ϕ	a	p	ϕ	a
0.05	2° 52′	2.00028	0.55	28 49	2.03364
0.10	5 43	2.00124	0.60	30 58	2.03976
0.15	8 32	2.00272	0.65	33 1	2.04635
0.20	11 19	2.00474	0.70	35 0	2.05339
0.25	14 2	2.00731	0.75	36 52	2.06087
0.30	16 42	2.01041	0.80	38 40	2.06879
0.35	19 17	2.01403	0.85	40 22	2.07711
0.40	21 48	2.01818	0.90	41 59	2.08585
0.45	24 14	2.02284	0.95	43 32	2.09497
0.50	26 34	2.02799	1.00	45 0	2.10447

Applying this third method to the examples already worked by the first method and the second method, we get for the range and time in the first example

$$x = 11335 \text{ ft.},$$
$$t = 21.51 \text{ sec.};$$

and for the coordinates and the time in the second example

$$x_2 = 22906 \text{ ft.},$$
$$y_2 = 266 \text{ ft.},$$
$$t_2 = 16.78 \text{ sec.}$$

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A THEOREM CONCERNING UNIFORM CONVERGENCE

By Geo. D. Birkhoff

In this note I aim to prove the converse of the theorem: **If the series**

$$|u_1(x)| + |u_2(x)| + \cdots \tag{1}$$

converges uniformly in the interval $a \leqq x \leqq b$, **then the series**

$$u_1(x) + u_2(x) + \cdots \tag{2}$$

converges uniformly in the same interval, however its terms may be rearranged.*

The converse theorem is the following:

Theorem: *If a series* (2) *is uniformly convergent in the interval* $a \leqq x \leqq b$ *however its terms may be re-arranged, then the absolute value series* (1) *is uniformly convergent in the same interval.*†

To show that series (1) is uniformly convergent, is to show that for every ϵ there is an N such that whenever $n \geqq N$,

$$R_n(x) < \epsilon$$

for every point x of the interval.‡

Here $R_n(x)$ denotes the remainder of the series (1) after the n^{th} term:

$$R_n(x) = |u_{n+1}(x)| + |u_{n+2}(x)| + \cdots;$$

and since all the terms of (1) are positive, $|R_n(x)| = R_n(x)$.

The proof is indirect, as follows. Suppose that under the given conditions the series (1) were *not* uniformly convergent; then for some fixed ϵ

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* For this theorem, and for a proof that the uniform and absolute convergence of (2) in some order does not suffice to ensure its uniform convergence in every order, see Bôcher, Annals of Mathematics, ser. 2, vol. 4 (1902–03), pp. 159–160.

† The question of the validity of this converse theorem was suggested to me by Professor E. H. Moore.

‡ This familiar definition of uniform convergence is here repeated at length because it is not always given correctly even in the best text-books. *Cf.*, for example, Picard, *Traité d'analyse*, vol. 1, 2d edition, 1901, p. 211; or Goursat, *Cours d'analyse*, vol. 1, 1902, p. 406.

there would be, for every m, some integer $n \geqq m$ and some point x_m of the interval, such that

$$R_n(x_m) > \epsilon.$$

Indeed, since $R_m(x_m) \geqq R_n(x_m)$, we may say that for every integer m there would be some point x_m such that

$$R_m(x_m) > \epsilon.$$

But since (2) is convergent for $x = x_m$, however its terms may be re-arranged, we know that the absolute value series (1) is convergent for $x = x_m$; so that we can find an integer $r > m$ such that $R_{m+r}(x_m) < R_m(x_m) - \epsilon$, whence $R_m(x_m) - R_{m+r}(x_m) > \epsilon$,
or

$$(3) \qquad |u_{m+1}(x_m)| + |u_{m+2}(x_m)| + \cdots + |u_{m+r}(x_m)| > \epsilon.$$

This inequality may be used as a recurrent formula for the determination of an infinite sequence of integers,

$$m_1, m_2, m_3, \cdots,$$

as follows: when $m = 0$, $r = m_1$; then $m_1 + r_1 = m_2$; $\cdots$; $m_i + r_i = m_{i+1}$, where the r_i corresponding to each m_i is so determined that (3) holds. These integers divide both the given series into sections or blocks, the i^{th} section, for example, containing the terms from the $(m_i + 1)^{\text{th}}$ to the $(m_{i+1})^{\text{th}}$, inclusive.

Consider a typical section of series (2), namely

$$(4) \qquad u_{m+1}(x_m) + u_{m+2}(x_m) + \cdots + u_{m+r}(x_m);$$

and let S'_m be the sum of the absolute values of the terms of one sign in this section, and S''_m the sum of the absolute values of the remaining terms (if any). Then from (3),

$$S'_m + S''_m > \epsilon,$$

so that one or other of the two quantities S'_m and S''_m must be $> \epsilon/2$; we may assume that

$$(5) \qquad S'_m > \frac{\epsilon}{2}.$$

Now re-arrange the terms of (4) in such a way that all the terms which be-

long to S'_m come first; and do the same in each section of the series. The result will be a certain re-arrangement of series (2) which may be denoted by

$$U_1(x) + U_2(x) + U_3(x) + \cdots. \tag{6}$$

We proceed to show that, contrary to hypothesis, this new series (6) cannot be uniformly convergent.

For, if the first p terms ($p \leqq r$) in a typical section of (6) are the terms which belong to S'_m, then, by (5),

$$| U_{m+1}(x_m) + U_{m+2}(x_m) + \cdots + U_{m+p}(x_m) | > \frac{\epsilon}{2} ;$$

that is, for every N there are two integers, m and $m + p$ (where $N < m < m + p$), and some point x_m, such that

$$| R'_m(x_m) - R'_{m+p}(x_m) | > \frac{\epsilon}{2} ,$$

where R' denotes the remainder in the series (6); and therefore either $| R'_m(x_m) |$ or else $| R'_{m+p}(x_m) |$ is $> \epsilon/4$.

Hence the supposition with which we started must be false, and the theorem is proved.

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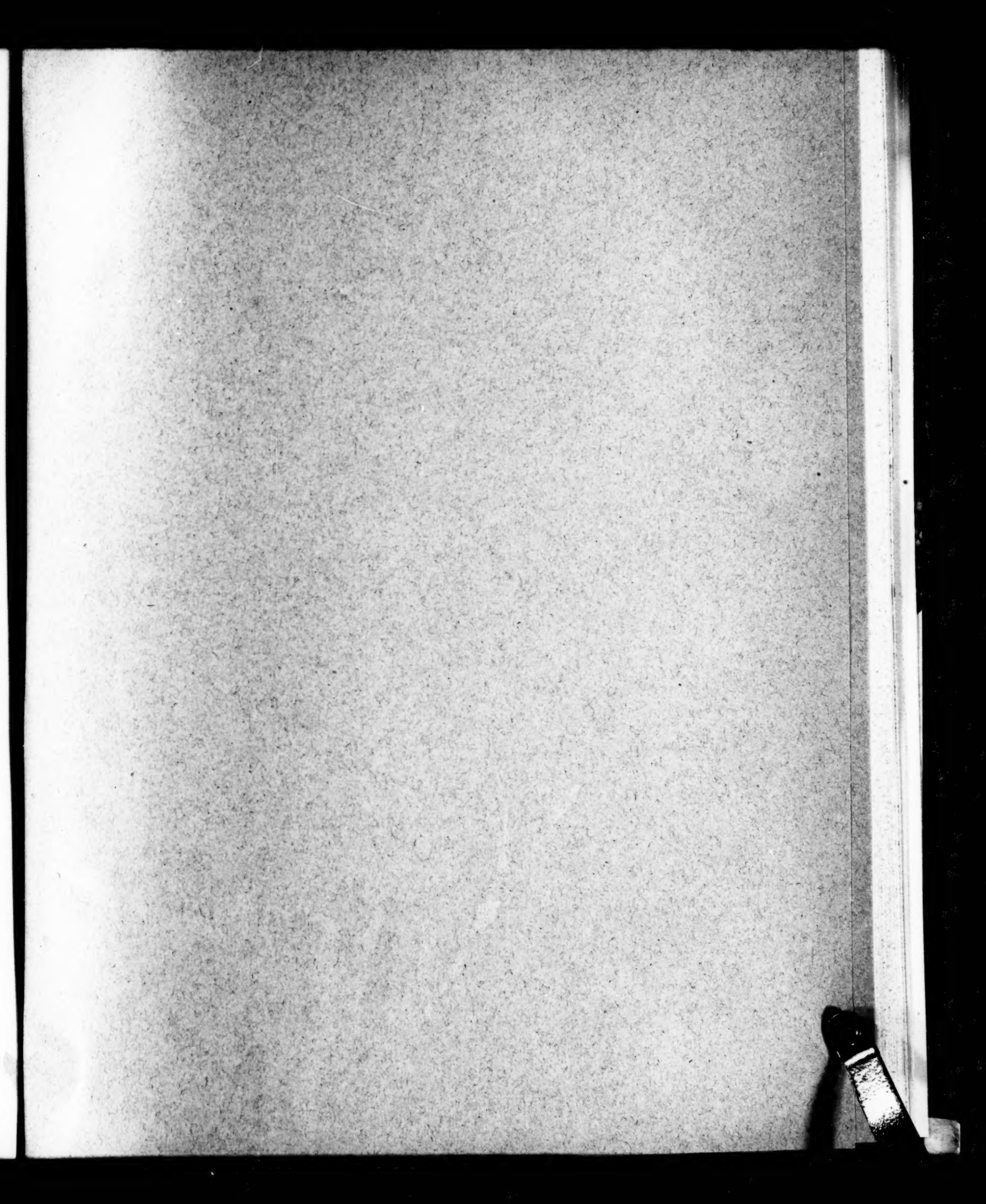

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